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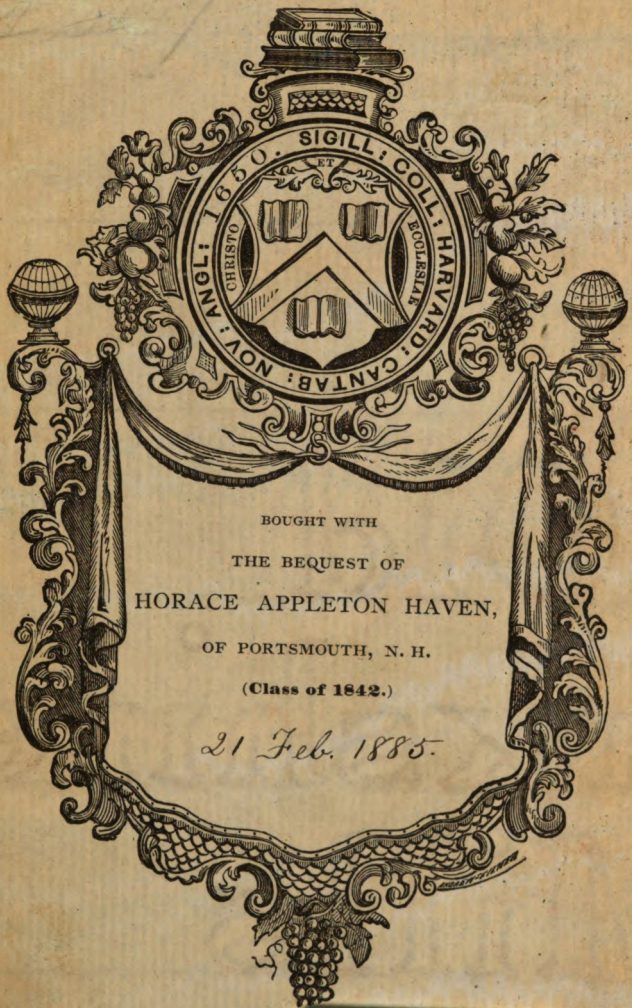

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SOLUTIONS OF PROBLEMS

GIVEN IN THE EXAMINATION FOR

THE JUNIOR MATHEMATICAL SCHOLARSHIP

IN THE YEARS 1866, 1867, 1868

BY

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QUESTIONS.

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1866.

I.

Algebra.

1. A man borrows P£ at r per cent. compound interest, and invests the sum in the r' per cent. stocks, the half-yearly dividends in which he immediately invests in the same stock. After the end of the fourth year, the stock rises in value 2 per cent. He then sells out, and, after repaying the loan with its accumulated interest, has Q£ left. But if the stock had not risen in value, he would have only just enough to repay the loan with its interest. Given P, Q, r, find the cost and percentage of the stocks in which he invested.

2. Eliminate x, y, z from

$$\frac{\frac{y}{z}+\frac{a}{x}}{y-z} = \frac{\frac{z}{x}+\frac{b}{y}}{z-x} = \frac{\frac{x}{y}+\frac{c}{z}}{x-y},$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab},$$

$$\frac{bc}{z^2}+\frac{ca}{x^2}+\frac{ab}{y^2} = \frac{z^2}{bc}+\frac{x^2}{ca}+\frac{y^2}{ab}.$$

3. Solve the equations:

(1) $x(x-1)(x-2)-a(a-1)(a-2) = 0;$

(2) $$\begin{cases} \dfrac{2ab}{x}+\dfrac{y}{a^2-b^2} = \dfrac{2ab}{a^2-b^2}, \\ \left(\dfrac{x}{a^2-b^2}\right)^2+\left(\dfrac{2ab}{y}\right)^2 = \left(\dfrac{2ab}{a^2-b^2}\right)^2; \end{cases}$$

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3. (*continued.*)

$$(3)\quad \begin{cases} (x+y)^{\frac{3}{2}} = -\dfrac{32}{65}xy, \\ \dfrac{65}{54}\left(1-\dfrac{y}{x}\right)^{\frac{1}{2}} = \dfrac{y}{y^{\frac{1}{2}}-x^{\frac{1}{2}}} - \dfrac{y}{y^{\frac{1}{2}}+x^{\frac{1}{2}}}; \end{cases}$$

(4) $x^2 - 29y^2 = \pm 5$ in positive integers.

4. If a series of terms in arithmetical progression is collected into groups of n terms, then the sums of these successive groups will form a series in arithmetical progression whose common difference is n^2 the original common difference.

5. Sum the series

$$2+4x+7x^2+12x^3+21x^4+38x^5+\ldots \text{ to } n \text{ terms,}$$

and find the value which x must not exceed in order that the nth term may be greater than all the following terms to infinity.

6. A binomial, having one term rational and the other a quadratic surd, when cubed and divided by 2, is equal to itself with its rational term doubled. Find the terms.

7. State and prove the multinomial theorem; and shew its application to the solution of the following problem:

What is the best number to back in throwing four dice; and on what other number are the odds the same as on eight?

8. There are ten counters in a bag marked with numbers. A person is allowed to draw two of them. If the sum of the numbers on these two is even, he receives as many shillings; if odd, he pays as many. Is the value of his expectation greater when the counters are numbered from 0 to 9 or from 1 to 10?

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9. If $N, a, b, c \ldots\ldots$ are positive integers, N not being a perfect square, and if

$$\sqrt{N} = a + \frac{1}{b+}\,\frac{1}{c+}\,\ldots\ldots\,\frac{n}{\sqrt{N}+m},$$

shew that m and n are integers.

10. If n is a positive integer, shew that

$$n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots\ldots = 1.2.3\ldots n.$$

11. If n is a prime number, prove that none of the coefficients in the expansion of $(a+b)^n$ are prime numbers except those of the first two terms and of the last two.

12. Prove the following properties of numbers:

(1) All cubes are of the form $9m$ or $9m \pm 1$;

(2) $n(n+1)(n+2)(5n^2-1)$ is divisible by 24.

13. If a_1', b_1', c_1', &c. are the minors of the determinant $(a_1 b_2 c_3)$, shew that

$$a_3' b_1' c_2' - a_2' b_3' c_1' = (a_1 b_2 c_3)\,\{a_3 b_1 c_2 - a_2 b_3 c_1\}.$$

14. The equation $x - \sqrt{a^2+x^2} = -\dfrac{na^2}{\sqrt{a^2+x^2}}$ is satisfied either by $x = \infty$ or $x = -a\dfrac{n-1}{\sqrt{2n-1}}$. But if the equation is solved the latter root only appears. Explain this.

15. Sum the series

(1) $1.3.2^2 + 3.5.3^2 + 5.7.4^2 + \ldots\ldots$ to n terms;

(2) $\dfrac{3.6}{5.8} + \dfrac{4.7}{12.15} + \dfrac{5.8}{21.24} + \ldots\ldots$ to n terms;

(3) $\dfrac{1}{2.3} + \dfrac{3.5}{3.4.3} + \dfrac{5.5^2}{4.5.3^2} + \dfrac{7.5^3}{5.6.3^3} + \ldots$ to n terms.

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II.

Trigonometry and Theory of Equations.

1. On what principles are trigonometrical ratios affected with signs? Give a general formula for all angles which have the same sine.

2. Prove that

(1) $$\frac{\tan A}{\tan A - \tan 3A} + \frac{\cot A}{\cot A - \cot 3A} = 1;$$

(2) $$(1+\sin A)^{\frac{1}{2}} = 1 + 2\sin\frac{A}{4}\left(1-\sin\frac{A}{2}\right)^{\frac{1}{2}};$$

(3) $$\sin 2A + \sin 2B + \sin 2C = 4\sin(B+C)\sin(C+A)\sin(A+B),$$

if $A+B+C = 360°$.

3. Discuss the ambiguous case in the solution of plane triangles: and if c is the ambiguous side of which c_1 and c_2 are the two values, prove that $c_1 c_2 = \pm(a^2 - b^2)$.

If $a = 235.74$, $b = 423.213$, $A = 57° 34' 25''$, solve the triangles.

4. If from any point P in a circle lines are drawn to the extremities A, B and to the point of contact C of a side of the circumscribing square, shew that

$$\frac{(1+\cot PCA)^2}{\cot PBA} = \frac{(1+\cot PCB)^2}{\cot PAB}.$$

5. If $\cos(a+\theta) = \cos a\cos\phi - \cos b\sin a\sin\phi$, and if θ and ϕ are small, shew that $\theta = \phi\cos b + \frac{\phi^2}{2}\cot a(\sin b)^2$, nearly.

6. Express $\tan n\theta$ in powers of $\tan\theta$; and expand $\cos n\theta$ in powers of $\cos\theta$.

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7. The triangle $A'B'C'$ circumscribes the escribed circles of the plane triangle ABC; shew that

$$\frac{B'C'}{a\cos A} = \frac{C'A'}{b\cos B} = \frac{A'B'}{c\cos C}.$$

8. A point is joined to the angular points of a polygon, whereby triangles are formed which have a common vertex, and have the sides of the polygon for their bases. For each triangle are drawn the inscribed circle, and the escribed circle which touches the base externally. Shew that the algebraic sum of the portions of the bases intercepted between the points of contact of these circles is zero.

9. Prove the following formulae of spherical trigonometry:

(1) $\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$;

(2) $\sin\frac{A}{2}\sin\frac{b+c}{2} = -\sin\frac{a}{2}\cos\frac{B-C}{2}$;

and find the radius of the small circle inscribed in a given triangle.

10. The angular points of a spherical triangle right-angled at C are joined by chords. If in the plane triangle thus formed C' corresponds to C, shew that

(1) $\cos C' = \sin\frac{a}{2}\sin\frac{b}{2}$;

(2) $C' = \frac{\pi}{2} - \frac{ab}{4r^2}$, if the triangle is small.

11. Prove that in an equation the greatest negative coefficient increased by unity is greater than the greatest root.

12. Determine the condition that $x^3 - px^2 + qx - r = 0$ should have two equal roots. Can you hence infer the condition that two roots should be impossible? Does your inference admit of generalization?

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13. Solve the equations :

(1) $x^3-3x-1=0$;

(2) $x^3-9x+28=0$;

(3) $x^4+8x^3+x^2-24x+9=0$;

(4) $5x^5-43x^4+60x^3+132x^2-65x+7=0$,

which is such that if one root is removed the sum of the combinations 2 and 2 together of the remaining four will be 26.

14. If a, b, c are the roots of $x^3-3m^2x-m^3=0$, form the equation whose roots are $\frac{b}{c}, \frac{c}{a}, \frac{a}{b}$.

15. Sum the following series :

(1) $\frac{x\sin\theta}{1}+\frac{x^2\sin 2\theta}{1.2}+\frac{x^3\sin 3\theta}{1.2.3}+\ldots\ldots$;

(2) $(1-\sqrt{-1})-\sqrt{-2}-(1+\sqrt{-1})-\sqrt{2}-\ldots\ldots$;

and shew that 7 is the least number of terms the sum of which is rational.

Also expand into a series of ascending powers of x

$$\frac{a+bx}{x^2-2x\cos\omega+1},$$

and give the general term of the series.

III.

Pure Geometry.

1. Describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

2. If straight lines are drawn bisecting the interior angles of a quadrilateral, shew that they form another quadrilateral whose diagonals pass through the intersections of the opposite sides of the first.

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3. Given the vertex and the centres of the inscribed and circumscribed circles of a triangle, construct the triangle.

4. If two straight lines be cut by parallel planes they shall be cut in the same ratio.

If three straight lines which do not all lie in the same plane are cut in the same ratio by three planes, two of which are parallel, shew that the third is in general also parallel to these two. State the exceptional case.

5. From a point on a diameter a tangent is drawn to a circle, and its point of contact is joined to the extremity of the diameter at right angles to the first. Shew that the tangent, the first diameter, and the joining line form an isosceles triangle.

6. The difference of the squares of the tangents from any point to two circles varies as the distance of the point from their radical axis.

7. A, B, C, D are fixed points in a given right line; find the locus of the point at which AB and CD subtend equal angles.

8. A pair of ordinates MP, $M'P'$ perpendicular to the major axis of an ellipse intersect the auxiliary circle in Q, Q'; shew that, if C is the common centre of the curves, the areas of the triangles CQQ', CPP' are proportional to the major and minor axes of the ellipse. Hence shew that the area of the triangle formed by conjugate semi-diameters and the line joining their extremities is constant.

9. Two sides of a parallelogram are fixed in position, and one vertex moves on a fixed straight line, shew that the envelope of one diagonal is a parabola.

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10. Prove the following properties of a pencil of lines a, b, c, d which are cut by any transversal in the points A, B, C, D;

(1) $\sin A \sin bc + \sin B \sin ca + \sin C \sin ab = 0$;

(2) $\sin ab \sin cd + \sin ac \sin db + \sin ad \sin bc = 0$;

(3) if $\frac{AD}{AB} : \frac{CD}{CB} = -1$, then $\frac{AC}{AD} : \frac{BC}{BD} = 2$.

11. If any line perpendicular to the axis of a hyperbola is terminated by the asymptotes, the rectangle under the segments into which it is divided by the curve is equal to the square of the semi-axis minor.

12. If perpendiculars are drawn from any point of a circle upon the sides of an inscribed triangle, the feet of the perpendiculars are in one straight line. Prove this and reciprocate it.

13. If two conics have double contact with each other, or if they have a contact of the third order, any tangent to the one is cut harmonically at the points where it meets the other and where it meets the chord of contact.

14. Two triangles intersect in the same conic; shew that they are co-polar.

IV.

Analytical Geometry.

1. Find the equation of a straight line in terms of trilinear coordinates; what is the geometrical meaning of the several constants?

2. Perpendiculars are drawn to the lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$, at the points where they cut the line

$$l\alpha + m\beta + n\gamma = 0;$$

find their equations, and explain the result when

$$l : m : n = \sin A : \sin B : \sin C.$$

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3. Shew that a straight line is represented by an equation of the first degree, when the coordinates of a point are the cotangents of the angles which the lines joining the point to two fixed points make with the line joining the fixed points; interpret the equations

$$x+y=0, \qquad xy+1=0.$$

4. If p, q, r are the perpendiculars from the vertices of a triangle upon the line joining the centres of the inscribed and circumscribed circles, and a, b, c the sides of the triangle, prove that

$$\frac{ap}{-a+b+c}+\frac{bq}{a-b+c}+\frac{cr}{a+b-c}=0.$$

5. The equations of two tangents to a parabola and of their chord of contact being given, find the equation of the tangent which is parallel to the chord.

6. Find the excentricity of an ellipse which is such that the vertex of an isosceles triangle whose base is the focal radius vector to the extremity of the minor axis, and vertical angle $= 120°$, will fall on the curve.

7. Two straight lines including a right angle pass through the vertex of a triangle. Find the locus of intersection of one of them with the perpendicular drawn to the base at the point where it is cut by the other.

8. Prove that $\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}=1$ expresses an ellipse and a hyperbola intersecting at (x, y) orthogonally and confocal with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

9. Find the condition that the general equation of the second degree should represent a curve passing through the origin and cutting both axes at right angles in the other points where it meets them.

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10. A person wishing to ascertain the dip of a stratum bores from a horizontal plane vertical holes at the three corners of a square whose side is h; he finds the depths to be respectively a, b, c; determine the dip, and the depth of the stratum at the fourth corner of the square.

11. If

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = xu_x + yu_y + u_1$$

is the equation of a central conic, shew that the area contained by a chord whose middle point is (x', y') and lines drawn from its extremities to the centre is

$$\frac{\{\nabla u' - (ab - h^2)u'^2\}^{\frac{1}{2}}}{ab - h^2},$$

∇ being the eliminant of $u_x = 0$, $u_y = 0$, $u_1 = 0$.

12. (r_2, a_2) (r_3, a_3) are the focal coordinates of points on a conic section, ρ_1 the focal distance of the pole of the chord joining them; shew that, if $2b$ is the minor axis of the curve,

$$\frac{1}{r_2 r_3} - \frac{1}{\rho_1^2} = \frac{1}{b^2}\left(\sin\frac{a_2 - a_3}{2}\right)^2.$$

13. Shew that the point of intersection (x, y) of normals to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ at the extremities of the chord whose pole is (x', y') is determined by the equations

$$\frac{x}{x'(b^2 - y'^2)} = \frac{-y}{y'(a^2 - x'^2)} = \frac{a^2 - b^2}{b^2x'^2 + a^2y'^2}.$$

Find the coordinates of the centre of curvature for any point (x', y').

14. Find the equation of the polar of any point $(\alpha', \beta', \gamma')$ with reference to the curve

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0,$$

and determine the coordinates of the centre of the curve, and the conditions that it should represent a circle.

15. Shew how to find the radical axis of two circles whose trilinear equations are given.

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V.

Differential Calculus.

1. Find the area of a small circle of a sphere; and if two equal small circles intersect each other, find the area common to both.

2. Find the rth derived function of $\epsilon^{ax}\cos nx$, and of $\frac{x}{x^2+a^2}$; and if $u = F(x, y) = 0$, find $\frac{d^2y}{dx^2}$ in terms of partial differentials of u. Apply the result to $x^3+y^3-6axy = 0$.

3. If $x^{\frac{1}{2}}$, and not x, is the independent and equicrescent variable, what are the equivalents of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$?

4. Assuming Taylor's theorem, prove the process for the evaluation of indeterminate quantities, and for the determination of orders of infinitesimals.

Evaluate $\left(2-\frac{\sin x}{x}\right)^{(\cot x)^2}$, when $x = 0$; and shew that $\left\{\frac{a_1{}^{nx}+a_2{}^{nx}+\ldots+a_n{}^{nx}}{n}\right\}^{\frac{1}{x}} = a_1 a_2 \ldots a_n$, when $x = 0$.

5. Expand $x\epsilon^{\cos x}$ in ascending powers of $\cos x$; and shew by the method of derivation that

$$(1+x)^{\frac{1}{x}} = e\left\{1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right\}.$$

6. Find $f(x)$, when $f(x+y)+f(x-y) = 2xf(x^2-y^2)$.

7. Investigate a process for finding the maxima and minima values of $f(x)$; and give geometrical illustrations; shewing the necessity of a change of sign in $f'(x)$ at the critical value.

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7. (*continued.*)

What is the geometrical line denoted by $f''(x)\,dx^2$?

Find the pair of conjugate diameters of an ellipse, the sum of which is a maximum.

8. If $u = F(\xi, \eta, \zeta)$,

and $\xi = x^2 - 2yz$, $\eta = y^2 - 2zx$, $\zeta = z^2 - 2xy$, shew that

$$2(x^3+y^3+z^3+xyz)\left(\frac{du}{d\xi}\right) = \left(\frac{du}{dx}\right)(x^2-yz) - \left(\frac{du}{dy}\right)(z^2+xy) - \left(\frac{du}{dz}\right)(y^2+zx).$$

9. Find the equation to the cycloid; prove that (the starting point being the origin) the normal is the chord of the generating circle, and that the radius of curvature is twice the normal. Hence find the evolute.

10. If $u=0$ is the equation to a curve of the second degree, to what lines are $\left(\frac{du}{dx}\right) = 0$, and $\left(\frac{du}{dy}\right) = 0$ the equations? Also what lines do they represent when u is the equation to a curve of the nth degree?

11. Trace the following curves:

(1) $y^2 = x^2 \dfrac{(x-a)\,(x-2a)}{(x+a)\,(x+2a)}$;

(2) $r = a \dfrac{\sin\theta}{\sin 3\theta}$;

(3) $r^2 + 2r(c - a\cos\theta) - c(2a\cos\theta - c) = 0$.

12. What means do you apply for determining the nature of a curve at a given point? What is the nature at the origin of the curve whose equation is

$$y^4 - 2x^2y^2 + x^4 + 2axy^3 - 5ax^2 = 0\,?$$

13. Define, and find the value of the radius of curvature of a plane curve. Shew that it is the radius of the circle passing through

$$(x, y),\ (x+dx, y+dy),\ (x+2\,dx+d^2x,\ y+2\,dy+d^2y).$$

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Find the locus of the extremities of lines drawn from a given point equal and parallel to the radii of curvature at the several points of the ellipse.

14. Find the curve in which the perpendicular from the origin on the tangent is one-half of the normal.

15. Find the envelope of the osculating circle of a parabola.

VI.

Problems.

1. Two lines of lengths a and b are drawn from the same point and are inclined at an angle θ; from their extremities perpendiculars are drawn to each of them: find the area of the parallelogram thus formed.

2. Two numbers end with the same digit; and when they are divided by 7, the quotient of each is the remainder of the other: find the numbers.

3. Ellipses are inscribed in a given parallelogram; shew that their foci lie on an equilateral hyperbola.

4. The sum of three terms of a geometric series is given. Find, without using the differential calculus, what the terms of it must be in order that their continued product may be a maximum.

5. If x is a number less than unity, then

$$\{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots\ldots\}^{-1} = (1+x)(1+x^2)(1+x^3)(1+x^4)\ldots\ldots,$$

the factors on each side being continued to infinity.

6. Shew that there is a point on the curve $y^3 = x^2 - x^3$, whose abscissa is $> \frac{9}{10}$ and < 1, such that the normal passes through the origin.

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7. In carrying out completely the process of finding the G. C. M. between two numbers which are prime to each other, the quotients in order are 5 three times and then 3 twice. Find the numbers.

8. From the origin of coordinates tangents are drawn to the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$. Shew that the coordinates of the point of contact of the tangent touching the nth branch are found by substituting for θ the value given by $\{(2n-1)\pi - \theta\} \tan\frac{\theta}{2} = 2$.

9. Find a fraction x and a whole number y such that $xy - \frac{y}{x}$ shall be the square of a whole number.

10. Into a vessel containing a given quantity of a fluid A, another fluid B flows at a uniform rate; the fluids being at once completely mixed flow out of the vessel at the same rate. Determine the time at which the vessel contains an nth part of the original quantity of the fluid A.

11. If in a series of numbers $a_1 b_1 c_1, a_2 b_2 c_2, \ldots a_n b_n c_n$, any group $a_r b_r c_r$ is connected with the preceding group $a_{r-1} b_{r-1} c_{r-1}$ by the equations

$$a_r a_{r-1} = b_{r-1} c_{r-1}, \quad b_r b_{r-1} = c_{r-1} a_{r-1}, \quad c_r c_{r-1} = a_{r-1} b_{r-1};$$

shew that $a_n = a^{\frac{1-(-2)^n}{3}} (bc)^{\frac{1-(-2)^{n-1}}{3}}$.

12. C is the centre of curvature of the point P of an ellipse. A perpendicular is drawn to CP through C. The part of this perpendicular cut off by the diameter through P is one-third of the radius of curvature of the evolute at C.

13. Bisect a spherical triangle having a right angle by an arc of a great circle drawn through one of the other angles.

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14. Find the equation to a circle which touches the lines $\alpha = 0$, $\beta = 0$, and the conic $\alpha\beta = k\gamma^2$.

15. Prove by the method of projections that of all polygons of a given number of sides circumscribing an ellipse, the one of least area is such that each side is parallel to the line joining the points of contact of adjacent sides.

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1867.

I.

Algebra.

1. If $b_1c_1+c_1a_1+a_1b_1=0$ and $a_2+b_2+c_2=0$, shew that

$$\frac{a_2b_1c_1}{b_1b_2-c_1c_2}+\frac{b_2c_1a_1}{c_1c_2-a_1a_2}+\frac{c_2a_1b_1}{a_1a_2-b_1b_2}=0;$$

and simplify

$$(a+b)\left\{a^2+b^2-ab+\frac{\sqrt{3}}{2}(a^2-b^2)\right\}^{\frac{1}{2}} -(a-b)\left\{a^2+b^2+ab-\frac{\sqrt{3}}{2}(a^2-b^2)\right\}^{\frac{1}{2}}.$$

2. Solve the equations:

(1) $\sqrt[3]{2a+x}+\sqrt[3]{2a-x}=\sqrt[3]{a}$;

(2) $\left.\begin{aligned} x(x^2-3y^2)&=\frac{a^3}{\sqrt{2}},\\ x^2+y^2&=a^2; \end{aligned}\right\}$

(3) $2x^4+x^3+2x^2+3x+18=0$;

(4) $\left.\begin{aligned} &a^2x+b^2y+c^2z=0,\\ &(b-c)^2yz+(c-a)^2zx+(a-b)^2xy=0,\\ &x^2+y^2+z^2=1. \end{aligned}\right.$

3. Eliminate a and b from $ax=by=\frac{a^2b^2}{a^2+b^2}$, and

$$m=\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}.$$

4. Investigate a formula for determining the number of divisors of a given number.

5. The number $p_0+10^3p_1+10^6p_2+10^9p_3+\ldots+10^{3m}p_m$ is divisible by 37, if $p_0+p_1+p_2+p_3+\ldots+p_m$ be divisible by 37.

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6. Shew that if a, b, c are unequal and $a+b+c>0$ then $a^3+b^3+c^3>3abc$, and that if x be real, then $\frac{x^2-x-5}{x-3}$ can have no real value between 3 and 7.

7. Sum the following:

(1) $1+x(x+2)+x^2(x^2+2x+2)+x^3(x^3+2x^2+2x+2)+\ldots$ ad inf.

(2) $\frac{1+2}{1.2}+\frac{1+2+3}{1.2.3}+\frac{1+2+3+4}{1.2.3.4}+\ldots\ldots$ ad inf.

(3) $\frac{19}{1.2.3}\cdot\frac{2}{3}+\frac{28}{2.3.4}\cdot\left(\frac{2}{3}\right)^2+\frac{39}{3.4.5}\left(\frac{2}{3}\right)^3+\frac{52}{4.5.6}\cdot\left(\frac{2}{3}\right)^4+\ldots$ to n terms.

8. When is it possible to solve an equation of the form $x^2-ay^2=b$ by developing $\sqrt{a}$ as a continued fraction? Solve $x^2-30y^2=1$.

9. Three members are to be elected into the Hebdomadal Council, and no elector can vote for more than two candidates. What is the least proportion of the whole constituency which a minority can be in to be able to return one member, (1) when an elector may give a candidate one of his votes, but need not vote for more than one; (2) when an elector may give both his votes to the same candidate?

10. Shew that $3n$ is the most probable number to throw with $2n$ counters, each marked with 1 on one face and 2 on the other; also shew that the chance of throwing $3(n+1)$ with $2(n+1)$ counters is less than the chance of throwing $3n$ with $2n$ counters in the ratio $\frac{2n+1}{2n+2}$.

11. Find two fractions such that the quotients of their sum by their product and of their difference by their quotient shall be both squares.

12. If $P_r=\frac{1.3.5\ldots\ldots(2r-1)}{2.4.6\ldots\ldots 2r}$, sum the series

$$P_{2n+1}+P_1P_{2n}+\ldots\ldots+P_nP_{n+1}.$$

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II.

Trigonometry and Theory of Equations.

1. Establish the following:

(1) $(\sin A)^6 + 3(\sin A)^2(\cos A)^2 + (\cos A)^6 = 1$;

(2) $\sec A + \sec(120° + A) + \sec(120° - A) + 3\sec 3A = 0$.

2. If A, B, C are the angles of a triangle,

$$\frac{\cos 3A}{\sin(A-B)\sin(A-C)} + \frac{\cos 3B}{\sin(B-C)\sin(B-A)} + \frac{\cos 3C}{\sin(C-A)\sin(C-B)} = 4.$$

3. If R, r, r_a, r_b, r_c are the radii of the circumscribed, inscribed, and escribed circles respectively of a triangle, and C one of the angles, then $4R\cos C = r + r_a + r_b - r_c$.

4. If D, E, F are the points in the sides of the triangle ABC where the perpendiculars from the opposite angles meet the sides BC, CA, AB respectively, and if O is the point of intersection of these perpendiculars, then

$$\frac{AO}{\cos A} = \frac{BO}{\cos B} = \frac{CO}{\cos C} = 4R' = \frac{r'}{\cos A \cos B \cos C},$$

where R' and r' are the radii of the circumscribed and inscribed circles of the triangle DEF.

5. On the sides a, b of a right-angled triangle as diameters circles are described; shew that the sum of the distances of their common tangents from the right angle is $\frac{(a+b)ab}{a^2+b^2}$.

6. The base BC of a triangle ABC is fixed, and the vertex A moves on the circumference of a circle which touches BC at

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its middle point A', and whose diameter is equal to BC; shew that

$$\sin 2AA'B = \frac{\sin(B-C)}{\sin(B+C)}.$$

7. Prove De Moivre's theorem; and shew that

$$\left(\sin\frac{\pi}{7}\right)^6+\left(\sin\frac{2\pi}{7}\right)^6+\left(\sin\frac{3\pi}{7}\right)^6+\left(\sin\frac{4\pi}{7}\right)^6+\left(\sin\frac{5\pi}{7}\right)^6+\left(\sin\frac{6\pi}{7}\right)^6=2\tfrac{3}{16}.$$

8. Find the continued product of

$$1+\sec A,\quad 1+\sec\frac{A}{2},\quad 1+\sec\frac{A}{2^2},\ \ldots\ldots\ 1+\sec\frac{A}{2^{n-1}};$$

and verify the result by equating A to zero.

9. In a spherical triangle shew that

(1) $$\cos A = \frac{\cos a-\cos b\cos c}{\sin b\sin c};$$

(2) $$\left(\cos\frac{a}{2}\right)^2 = \frac{\cos\frac{1}{2}(A-B+C)\cos\frac{1}{2}(A+B-C)}{\sin B\sin C}.$$

10. Solve the equations:

(1) $x^3+6x^2+49=0$;

(2) $x^3-9x-\sqrt{54}=0$;

(3) $x^4+4x^3+5x^2+2x-6=0$;

(4) $x^4-9x^3+81x+81=0$.

11. If x_1, x_2, x_3 are the roots of $x^3-3px+q=0$, then

$$\Sigma\{(x_2+x_3)^2-2x_1(x_3+x_1)(x_1+x_2)\}=6(p+q).$$

12. Transform the equation $x^3-px^2+qx-r=0$ into one, the roots of which are the squares of the differences of the roots of the given equation. Hence shew that the given equation has imaginary roots if $p^2=3q$.

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III.

Geometry.

1. Similar triangles have to one another the duplicate ratio of their homologous sides.

2. Through any point O in AA' the perpendicular on the hypothenuse of a right-angled triangle ABC, OB_1, OC_1, B_2OC_2 are drawn parallel to AB, AC, BC respectively meeting AC in B_1B_2, and AB in C_1C_2: shew that B_1B_2 has to C_1C_2 the triplicate ratio of AC to AB.

3. If a solid angle is contained by three plane angles, any two are together greater than the third.

4. Similar polygons inscribed in circles are to one another as the squares of their diameters.

5. The perpendiculars of a triangle ABC meet the opposite sides in A', B', C'; shew that the circles described about the triangles $AB'C'$, $BC'A'$, $CA'B'$ are together equal to 8 times the semicircles described on the perpendiculars from the centre of the circumscribing circle upon the sides.

6. The radical axis of two circles having each of them similar contact with two others, passes through their external centre of similitude.

7. The locus of the intersections of lines drawn from the extremities of a variable diameter of a circle to two fixed points on the circle is a circle whose centre is the intersections of the tangents at the fixed points.

8. Prove that the rectangle contained by the distances of the foci from any tangent of an ellipse is constant.

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9. Given in position one focus and a tangent, and in length the transverse axis of a hyperbola; find the locus of the other focus.

10. A pencil of lines $O.ABCD$ is cut by a transversal in $ABCD$. Shew that

(1) $BC \cot OAD + CA \cot OBD + AB \cot OCD = 0$;

(2) $\sin AOB \sin COD + \sin AOC \sin DOB + \sin AOD \sin BOC = 0$.

11. Deduce a property of the parabola by reciprocating the following theorem: lines joining the extremities of a pair of diameters of a circle are parallel.

12. Given a conic and a right line; shew how to project the conic into a circle, and the right line into the line at infinity. Hence generalize the theorem that a chord of one circle which is also a tangent to a concentric circle is bisected at the point of contact.

13. From any point P in a hyperbola, PN, Pn are drawn perpendicular to the axes, and PM, Pm to the asymptotes; prove that Nn and Mm are perpendicular to each other.

IV.

Algebraical Geometry.

1. Shew that the area of the triangle contained by the lines

$$(1+bc)x+(1-c)\ y+(1+b)\ (2c)^{\frac{1}{2}} = 0,$$

$$(1+c)\ x+(1+ca)y+(1-a)\ (2c)^{\frac{1}{2}} = 0,$$

$$(1-b)\ x+(1+a)\ y+(1+ab)(2c)^{\frac{1}{2}} = 0,$$

is $a+b+c+abc$.

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2. Find in terms of polar coordinates the equation of the chord of a conic, the focus being the pole; and its envelope when the angle subtended by it at the focus is constant.

3. What do the equations of a straight line, a circle, and of a conic respectively mean when the point (x, y) in terms of which they are expressed is not on the curve?

4. Find the locus of the intersection of two tangents to an ellipse the product of the perpendiculars on which from the centre of the ellipse is equal to the product of the semi-axes.

5. Prove that the lengths of tangents drawn to an ellipse from a given point are to one another as the diameters to which they are parallel.

6. Discuss the following:

(1) $12x^2-7xy-12y^2-35x+5y=0$;

(2) $x^2-10xy+y^2-4x-4y-2=0$;

(3) $\frac{1}{r}=2+\cos\theta+\sqrt{3}\sin\theta$.

7. If $ax^2+2hxy+by^2+1=0$ represent a hyperbola, then its equation referred to its asymptotes, as coordinate axes, will be $4(h^2-ab)xy=\sqrt{(a-b)^2+4h^2}$

8. Normals are drawn to an equilateral hyperbola from a given point; shew that the sum of their squares is equal to three times the square of the distance of the given point from the centre of the hyperbola.

9. Lines are drawn from the internal centre of similitude of two circles to the points of contact of an external common tangent; shew that if α, β, γ are the angles which these lines and the tangent make with the line joining the centres of the circles, then $\cot\alpha+\cot\beta+2\tan\gamma=0$.

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10. If from a fixed point A chords are drawn to a conic intersecting it in B, B', and a fourth point A' is taken so that the anharmonic ratio of $ABB'A'$ is constant, shew that the locus of A' is a conic having double contact with the given conic.

11. Normals at the extremity of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ intersect the polar of the focus and each other at points whose ordinates are y_1, y_2, y_3; y_1', y_2', y_3' are similar ordinates for a focal chord at right angles to the first; shew that

$$\left(\frac{y_1+y_2-2y_3}{y_1-y_2}\right)^2+\left(\frac{y_1'+y_2'-2y_3'}{y_1'-y_2'}\right)^2=\frac{a^2}{a^2-b^2}.$$

12. Determine the condition that the general equation of the second degree in trilinear coordinates may represent an equilateral hyperbola.

Equilateral hyperbolas are described about a given triangle; then the locus of their centres is the circle that passes through the middle points of the sides of the triangle.

V.

Differential Calculus.

1. If $\frac{6}{y}=2+\sqrt{22}\cos x$, then $\frac{d^2y}{dx^2}=y^3+y^2-y$.

2. Obtain the equation equivalent to

$$x^4\frac{d^2u}{dx^2}+2x^2(x+1)\frac{du}{dx}+u=0$$

when $\frac{1}{x}$ is the independent and equicrescent variable.

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3. Enunciate and prove Maclaurin's theorem. Expand by it y in ascending powers of x when

(1) $y = (\sin^{-1}\sqrt{x})^2$;

(2) $y^3 - 3xy^2 + x^3 = 0$.

4. Evaluate $\left(\frac{x}{a}\right)^{\frac{1}{\sqrt{a^2-x^2}}}$, when $x = a$; and shew that if

$a+b+c = 0$ and $x = 0$, $\dfrac{\tan ax + \tan bx + \tan cx}{\tan ax \tan bx \tan cx} = 1$.

5. Two similar ellipses are such that the major axis of the one coincides with the minor axis of the other both in length and direction; shew that the maximum chord of the larger ellipse which touches the other is equal to the line joining the extremities of the axes of the larger ellipse.

6. If $u = f(l, m, n)$, and

$$l = \frac{y-z}{x}, \quad m = \frac{z-x}{y}, \quad n = \frac{x-y}{z}, \quad \text{then}$$

$$(1+mn)\left(\frac{du}{dx}\right) + (1-n)\left(\frac{du}{dy}\right) + (1+m)\left(\frac{du}{dz}\right) = 0.$$

7. Define order and class of a given curve; if the order of a curve is n, what is its class? Find the first polar curve of $x^3+y^3-6axy = 0$.

8. Trace (1) $y^3 = \dfrac{x^4}{2x-a}$;

(2) $r^2 = 2a^2 \sec 2\theta \tan \theta$.

9. When x is equicrescent,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

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10. The radius of curvature at any point of

$$9a(a-x)^2 = (a+y)^2(a-2y)$$

varies as the square of the length of the normal intercepted between the point and the axis of abscissae.

11. The involute of a circle rolls along a given straight line; prove that the centre of the circle will trace out a parabola.

12. Find the envelope of the radical axis of two circles of given radius, the centre of one being fixed and the centre of the other being always upon a fixed right line.

VI.

Problems.

1. If each of two pairs of the equations

$$ax^3+bx+c = 0,$$
$$bx^3+cx+a = 0,$$
$$cx^3+ax+b = 0,$$

have a common root, shew that the third pair has also a common root; and if they all have a common root,

$$a^3+b^3+c^3-3abc = 0.$$

2. Two chords of a circle intersect at right angles. If the length of one of them be to the radius as $\sqrt{2} : 1$, then the difference of the segments of the other is equal to the former.

3. If $A+B+C = (2n+1)\pi$, shew that

$$\begin{vmatrix} \sin A, & \sin B, & \sin C \\ \cos A, & \cos B, & \cos C \\ \sin 3A, & \sin 3B, & \sin 3C \end{vmatrix} = 0.$$

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4. In a spherical triangle, if

$$\cos a \cos \frac{c}{2} = \cos\left(b + \frac{c}{2}\right), \text{ then } A = 2B.$$

5. If from any point without a triangle ABC a tangent is drawn to the inscribed circle of which the radius is r, and α, β, γ are the portions of it intercepted opposite to the angles A, B, C respectively, then

$$\alpha^2 \tan \frac{A}{2} + \beta^2 \tan \frac{B}{2} + \gamma^2 \tan \frac{C}{2} = \frac{\alpha\beta\gamma}{r}.$$

6. PQ, PR are tangents to a parabola, such that $PQ.PR$ varies as SP, S being the focus; then the locus of P is a parabola having the same parameter as the given parabola.

7. Through the extremity of the major axis of an ellipse a chord AP is drawn; the line joining the focus of the ellipse remote from A to the pole of AP cuts AP in Q; and a line through Q parallel to the minor axis meets the curve in R: shew that $(\cos ASQ)^2 = \cos ASR$.

8. A parabola and its circle of curvature at any point have in general only one chord of intersection. Prove this, and shew that the points where such chords are met by perpendiculars from the vertex lie in a cissoid.

9. If ω is the angle between the axes of coordinates, then the equation of the directrix of the parabola

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$

is $x(a + b\cos\omega) + y(a\cos\omega + b) = ab\cos\omega$.

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10. A given ellipse slides between two given straight lines at right angles to each other; shew that either focus will trace out the curve

$$4(a^2-b^2) = \left(\frac{b^2-x^2}{x}\right)^2 + \left(\frac{b^2-y^2}{y}\right)^2,$$

$2a$, $2b$ being the axes of the ellipse, and the given straight lines the axes of coordinates.

11. The focal distances of three points of a parabola are in arithmetic progression; shew that the normals at these points intersect in a semi-cubical parabola.

12. Find $f(x)$ when $f(x)f(y)-f(xy) = x+y$, x and y being independent.

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I.

Algebra.

1. Prove that

$$(x+y+z)^{2n+1}-x^{2n+1}-y^{2n+1}-z^{2n+1}$$

is divisible by $(y+z)(z+x)(x+y)$.

2. Extract the square root of

(1) $(\sqrt{3}+\sqrt{2})^2-4\sqrt[4]{6}(\sqrt{3}-\sqrt{2})$;

(2) $2a+2\sqrt{a^2-b^2-c^2+2bc}$.

3. Solve the equations:

(1) $$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=2,$$
$$\sqrt{\frac{x}{z}}+\sqrt{\frac{z}{x}}=3,$$
$$\sqrt{\frac{y}{z}}+\sqrt{\frac{z}{y}}=4\sqrt{x};$$

(2) $$\frac{16x^4-8x^2+1}{8x^3+4x^2+2x+1}=x-\frac{1}{2};$$

(3) $$(2x-1)(x+6)(x-9)+2(x-1)(x+9)(x-12)$$
$$-(2x-3)(x+15)(x-6)=0;$$

(4) $$\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}=a\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}};$$

(5) $2x^3+27x^2=54$.

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4. State and prove the rule for converting a circulating radix fraction in any scale of notation into a vulgar fraction.

Express $\frac{1}{3}$ as a radix fraction, 7 being the radix.

5. Expand $\frac{x}{\sqrt{(1+x)}}$ in a series proceeding by ascending powers of $\frac{x}{1+x}$.

6. Sum the following series to n terms:

$$1+x+x^2+2x^3+2x^4+5x^5+4x^6 \ldots,$$

$$\frac{3}{5.8.11}+\frac{6}{8.11.14}+\frac{9}{11.14.17}+\ldots.$$

7. Find the number of ways in which m things can be distributed into n compartments, n being less than m, and it being understood that no compartment is to be left altogether empty.

8. The series of pound weights by which the greatest number of pounds continuously can be weighed is 1 lb., 3 lbs., 9 lbs., 27 lbs., etc.

9. Determine the number of numbers less than a given number N and prime to it.

How many roots has the equation $x^N = 1$, which are not roots of unity of any lower denomination than N?

10. If the equation $x^2 - Ny^2 = -1$ is resoluble in integral numbers, $\frac{x}{y}$ is a convergent to $\sqrt{N}$.

Also given the least solution of the equation, shew how to find all the solutions. *Example;* $x^2 - 13y^2 = -1$.

11. Prove the following:

(1) The 18th power of any number not divisible by 3 is of the form $27m+1$;

(2) If n be any integer, $3^{2n} - 8n - 1$ is divisible by 64.

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12. Find the series equivalent to the product

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16}) \ldots \text{ ad inf.}$$

13. If $c_0, c_1, c_2 \ldots\ldots c_n$ be the coefficients in the expansion of $(1+x)^n$, shew that

$$\frac{c_0}{1.2}+\frac{c_1}{2.3}+\frac{c_2}{3.4}+\ldots\ldots+\frac{c_n}{n+1.n+2}=\frac{2^{n+2}-n-3}{(n+1).(n+2)}.$$

14. Find the sum of n terms of the series whose nth term is p times the sum of all the preceding terms. To what practical use is this series put?

15. A man has left his umbrella in one of three shops, which he visited in succession. He is in the habit of leaving it, on an average, once in every four times that he goes to a shop. Find the chance of his having left it in the first, second, and third shops, respectively.

II.

Trigonometry and Theory of Equations.

1. Establish the following:

(1) $16 \cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = 1$;

(2) $$\tan^{-1}\left(\sqrt{\frac{3}{2}}-1\right)+\tan^{-1}\frac{1}{6}\sqrt{\frac{3}{2}}+\tan^{-1}\left(\sqrt{\frac{3}{2}}+1\right)=\frac{2r+1}{2}\pi.$$

2. If D, d be the diameters of the circumscribed and inscribed circles of a regular polygon of which a is a side, then $D^2-d^2=a^2$.

3. If a, b, c be the sides of a triangle opposite to the angles A, B, C, and r, r_a, r_b, r_c the radii of the inscribed and escribed circles, then

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(1) $$\frac{\cos\frac{B-C}{2}}{\cos\left(60°-\frac{B}{2}\right)\cos\left(60°-\frac{C}{2}\right)} + \frac{\cos\frac{C-A}{2}}{\cos\left(60°-\frac{C}{2}\right)\cos\left(60°-\frac{A}{2}\right)} + \frac{\cos\frac{A-B}{2}}{\cos\left(60°-\frac{A}{2}\right)\cos\left(60°-\frac{B}{2}\right)} = 4;$$

(2) $$2r = \frac{a\tan\frac{B}{2}\tan\frac{C}{2} + b\tan\frac{C}{2}\tan\frac{A}{2} + c\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}};$$

(3) $$1 = \frac{ab}{(r_b+r_c)(r_c+r_a)} + \frac{bc}{(r_c+r_a)(r_a+r_b)} + \frac{ca}{(r_a+r_b)(r_b+r_c)}.$$

4. Given two sides of a triangle a, b and the angle A, solve it, a being less than b.

Shew also that the difference of the two values found for the third side is $2a\cos B$.

5. If the sides of a triangle be represented by

$$\frac{x}{y}+\frac{y}{z}, \quad \frac{y}{z}+\frac{z}{x}, \quad \frac{z}{x}+\frac{x}{y},$$

then the area $= \sqrt{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$.

6. Expand $\tan^{-1}x$ in terms of ascending powers of x, and shew that

$$1^2+2^2+\frac{3^2}{1.2}+\frac{4^2}{1.2.3}+\ldots\ldots = 5e,$$

where e is the base of Napierian logarithms.

7. Prove that

$$1-3\cos\theta+5\cos 2\theta-7\cos 3\theta+\ldots$$
$$\ldots+2\,(3\sin\theta-5\sin 2\theta+7\sin 3\theta-\ldots\ldots)^2 = 0,$$

both series being taken to infinity.

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8. If a, b, c be the sides of a spherical triangle opposite to the angles A, B, C, and R the radius of the circumscribed circle, then

$$(1)\quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = 2\tan R \cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2};$$

$$(2)\quad \cot A = \frac{\cos b \cos c - \cot B \cot C}{\cos b \cot B + \cos c \cot C};$$

also deduce the corresponding properties of a plane triangle.

9. If x_1, x_2, x_3, x_4 be the roots of

$$x^4 - ax^3 + bx^2 - abx + d = 0, \text{ then}$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2.$$

10. Solve the equation $x^3 - qx - r = 0$, when q and r are positive, and $27r^2$ less than $4q^3$.

11. Solve

(1) $x^3 - 12x - 8 = 0$;

(2) $x^3 + 9x + 6 = 0$;

(3) $16x^4 - 96x^3 + 144x^2 - 81 = 0$;

in (3) there are two equal roots.

III.

Pure Geometry.

1. State and explain Euclid's definition of the sameness of ratios.

2. The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides.

Give Euclid's proof of this theorem. Shew also that it can be deduced from the identical equation

$$AB.CD + AC.DB + AD.BC = 0,$$

connecting the distances between four points on the same straight line.

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3. Draw a straight line perpendicular to a plane from a given point above it.

4. Determine the circle which shall pass through a given point, and with respect to which a given point and straight line shall be pole and polar.

5. A system of circles has the same radical axis, but this axis does not meet any of them. Shew how to find the circle of the system which passes through a given point.

6. Prove the following properties of the parabola:

(1) The locus of the feet of the perpendiculars let fall from the focus on the tangent is the tangent at the vertex;

(2) A circle circumscribing a triangle circumscribed to a parabola passes through its focus.

7. Shew that in a hyperbola (1) the area contained by the asymptotes and any tangent is constant; (2) the conjugate diameters of the hyperbola are pairs of harmonic conjugates of the asymptotes.

8. Pairs of tangents drawn to an ellipse from any point outside it make equal angles with the focal radii vectores of the point.

9. Given in an ellipse a focus, a tangent and its point of contact, and a tangent; find the centre, the other focus, and the vertices of the ellipse.

10. AX and BY are equal arcs of a circle measured in opposite directions from two fixed points A and B on the circumference. If P and Q are two other fixed points on the same circumference, prove that the locus of the intersection of PX, QY is an equilateral hyperbola; and determine the directions of its asymptotes.

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11. $ABCD\ldots$, $A'B'C'D'\ldots$ are two homographic divisions upon the same straight line. Shew that if

$$[AA'BC] = [A'AB'C']$$

the two divisions are in involution.

12. Shew that two involutions on the same line, of which both have not real double points, always have a segment in common.

13. The polars of any point P with respect to a system of circles having the same radical axis, all pass through the opposite extremity of the diameter of the orthogonal circle passing through P.

Prove this theorem, and reciprocate it with respect to a circle having its centre at P.

14. Tangents from the vertices of a triangle ABC are drawn to a conic section, and meet the opposite sides in the points $a_1\ a_2$, $b_1\ b_2$, $c_1\ c_2$ respectively. Prove that a conic section can be drawn through these six points.

15. A system of conics passes through four fixed points, through two of which a conic C is drawn. Prove that the chord of intersection of C with any conic of the system continually passes through a fixed point.

IV.

Analytical Geometry.

1. Find the condition that the two straight lines

$$Ax+By+C=0, \text{ and } A'x+B'y+C'=0,$$

may be at right angles, the angle between the axes being ω.

2. Under what conditions will $\left(\frac{x}{a}+\frac{y}{b}-1\right)^2 = 2kxy$ represent a circle? If ω be the inclination of the axes, shew that the radius $= a\tan\frac{\omega}{2}$.

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3. Prove analytically the harmonic properties of the complete quadrilateral.

4. The directrix of the parabola

$$(\sqrt{a}x + \sqrt{c}y)^2 + 2dx + 2ey + \frac{d^2+e^2}{a+c} = 0$$

is given by $\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{c}} = 0$; and the focus by

$$\frac{x}{d} = \frac{y}{e} = -\frac{1}{a+c}.$$

5. Investigate the conditions that the general equation of the second degree between x and y, should represent (1) two straight lines, (2) two parallel straight lines, (3) two coincident straight lines.

6. If from any point in the plane of an ellipse a perpendicular be let fall upon its polar, the two confocal conics which can be drawn through the foot of the perpendicular, will touch at that point the one the polar, the other the perpendicular.

7. Normals are drawn to a parabola from a given point, and perpendiculars are let fall on them from the focus; shew that the sum of the squares of the perpendiculars $= 2(x-m)(x-2m)$, x being the abscissa of the given point measured from the vertex, and $4m$ the latus rectum.

8. From a given point as centre circles are described cutting an ellipse. Shew that the locus of the middle points of the common chords of the circles and the ellipse is an equilateral hyperbola, passing through the feet of the normals drawn from the given point to the ellipse.

9. Obtain the equation of the circle that passes through any three points of a hyperbola. Also deduce the equation of the evolute, and the length of the radius of curvature.

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10. Perpendiculars from the angles of a triangle meet the opposite sides in D_1, D_2, D_3; if O_1, O_2, O_3 be the centres of the escribed circles, shew that $D_1 O_1$, $D_2 O_2$, $D_3 O_3$ meet in a point which lies in the straight line joining the centres of the inscribed and circumscribed circles.

11. Prove that the equation of the polar of the point $\alpha' \beta' \gamma'$, with reference to the conic $F = 0$, is

$$\alpha' \frac{dF}{d\alpha} + \beta' \frac{dF}{d\beta} + \gamma' \frac{dF}{d\gamma} = 0.$$

12. Find the condition that

$$l\alpha + m\beta + n\gamma = 0$$

may touch the conic $F = 0$.

13. Tangents are drawn to the conic $LM = R^2$, at the points where it is intersected by the lines

$$L = \mu R, \qquad L = \mu' R.$$

What is the locus of the intersection of these tangents, if

$$a\mu + b\mu' + c = 0?$$

14. If the coordinates of a straight line be the reciprocals of its intercepts on two rectangular axes, find the tangential equation of a circle, of which the origin is the centre.

V.

Differential Calculus.

1. When is a function of a variable said to be continuous? What are the values of x for which the following functions or their derived functions become discontinuous?

$$\sqrt{\frac{10}{x} - \frac{10}{x+1} - \frac{3}{x+2}},$$

$$\sec^{-1}(x^2 - 3\pi x + 2\pi^2).$$

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2. Eliminate x from the equation

$$\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} + (x^2-1)\frac{dy}{dx} = 0,$$

given $x = \sin\theta + \cos\theta$;

and x, y, z from the equation

$$\frac{1}{x}\left(\frac{du}{dx}\right) + \frac{1}{y}\left(\frac{du}{dy}\right) + \frac{1}{z}\left(\frac{du}{dy}\right);$$

given $x = r\sin\phi\cos\theta$;

$y = r\sin\phi\sin\theta$;

$z = r\cos\phi$.

3. Prove Taylor's theorem.

Find an expression for the remainder after n terms of the expansion of $\epsilon^x \cos x$, and apply it to prove that the expansion is convergent for all values of x.

4. Expand $\sin\left(\frac{\pi}{2}\epsilon^x\right)$ in ascending powers of x.

5. If $\phi(a) = 1$, and $\psi(a) = \infty$, in what cases is the limit of the expression $[\phi(x)]^{\psi(x)}$, for $x = a$, equal to 1, less than 1, and greater than 1?

Evaluate

$$\frac{a\cos\theta + b\sin 2\theta}{\epsilon^m - \epsilon^{\frac{2m\theta}{\pi}}}, \text{ when } \theta = \frac{\pi}{2};$$

$$\frac{\sqrt[n]{x+a} - \sqrt[n]{y+a}}{\sqrt[m]{x} - \sqrt[m]{y}}, \text{ when } x = y;$$

$[\sin(1-x)]^{1-x}$, when $x = 1$.

6. Find the maximum and minimum values of the sum of the nth powers of conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

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7. The curves

$$(x^2+y^2)^2 = a^2(x^2-y^2),$$
$$(x^2-y^2)^2 = a^2(x^2+y^2),$$

have singular points at an infinite distance from the origin; investigate the nature of these points.

8. Find the equation to the tangent to the curve

$$(a+b)\,y = \frac{a^3}{x-a} - \frac{b^3}{x-b}$$

at its point of inflexion.

9. Find the equation to the normal of a curve referred to polar coordinates; and apply it to determine the polar coordinates of the point of intersection of normals at the extremities of a focal chord of a parabola.

10. Trace the curves

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{a}},$$
$$r = \epsilon^{\cot\theta};$$

and determine the motion of the singular points of the curve

$$(ay-x^2)^3 = a(xy-a^2)^3.$$

11. Shew that $\rho = r\dfrac{dr}{dp}$.

If ρ_1, ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola, of which $4m$ is the latus rectum, then $(\rho_1\rho_2)^{\frac{2}{3}} = (2m)^{\frac{2}{3}}(\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}})$.

12. Shew that the cycloid is the envelope of the curves obtained by the giving different values to a in

$$y = ax + 2c\cot^{-1}a.$$

13. If a series of curves have the same evolute, they all have the same number of points of inflexion, which lie upon an equal number of straight lines.

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VI.

Problems.

1. If $2s = a+b+c+d, \qquad 2\sigma = a+\beta+\gamma+\delta,$

shew that

$$(s-a)(\sigma-a)+(s-b)(\sigma-\beta)+(s-c)(\sigma-\gamma)+(s-d)(\sigma-\delta) = a\alpha+b\beta+c\gamma+d\delta.$$

2. Any polygonal number of order r, multiplied by $8\frac{r-2}{(r-4)^2}$, and increased by unity, makes a square number.

3. The sides of a triangle are such that

$$\frac{a}{1+m^2n^2} = \frac{b}{m^2+n^2} = \frac{c}{(1-m^2)(1+n^2)};$$

shew that the angles opposite to a and b are

$$2\tan^{-1}\frac{m}{n} \text{ and } 2\tan^{-1}mn,$$

and that the area is $\frac{mn}{m^2+n^2}bc$.

4. The nth fraction convergent to

$$\frac{1}{2\cos\theta-}\ \frac{1}{2\cos\theta-}\ \frac{1}{2\cos\theta-}\ \cdots\cdots$$

is $\frac{\sin n\theta}{\sin(n+1)\theta}$.

5. Shew that

$$\left\{\frac{1}{\cot\theta+}\ \frac{1}{\tan\theta+}\ \frac{1}{\cot\theta+}\cdots\right\}\left\{\tan\theta+\frac{1}{\cot\theta+}\ \frac{1}{\tan\theta+}\ \frac{1}{\cot\theta+}\cdots\right\} = \tan^2\theta.$$

6. If AA_1, BB_1, CC_1 be arcs of great circles bisecting the angles of a spherical triangle, and meeting the opposite sides

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in A_1, B_1, C_1, and R the radius of the circumscribed circle, then

(1) $$\sin AA_1 \tan R = 2 \sec \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2};$$

(2) $$\frac{\cos B - \cos C}{\cos \frac{A}{2}} \cot AA_1 + \frac{\cos C - \cos A}{\cos \frac{B}{2}} \cot BB_1 + \frac{\cos A - \cos B}{\cos \frac{C}{2}} \cot CC_1 = 0:$$

deduce the corresponding properties of a plane triangle.

7. $P_1 SP_2$ is a focal chord of a conic of which S is the focus and A the vertex; AP_1, AP_2 meet the latus rectum in Q_1, Q_2; then half the latus rectum is a mean proportional between SQ_1 and SQ_2.

8. If c_1, c_2 be the lengths of the normal chords at the extremities of a focal chord of a parabola of which l is the latus rectum, then $(c_1^{-2} + c_2^{-2})^3 = (l c_1 c_2)^{-2}$.

9. Pairs of tangents to the cissoid,

$$y^2 = \frac{x^3}{2a - x},$$

including a right angle, intersect in an ellipse.

10. The radius of curvature of the curve $y = \log \sec x$ is equal to $\frac{N}{y}$, where N is the length of the normal cut off by the axis and y the ordinate.

11. A straight line moves between two fixed straight lines at right angles to each other; if the middle point of the straight line lie on the curve $\left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 = 1$, then the straight line will always touch an ellipse.

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12. If α, β, γ be the angles subtended at any point by the sides of a triangle, find the loci represented by

$$(1) \quad \cos\alpha + \cos\beta + \cos\gamma = 1;$$

$$(2) \quad \cos\alpha + \cos\beta + \cos\gamma = -1.$$

13. A rectangular piece of paper $ABCD$ is turned over so that A coincides with C, and B with D, and is then flattened out. Shew that it is possible that the figure thus formed may be a regular hexagon; and determine the condition that this should be so.

14. The harmonic triangle of a quadrangle is given, and one of the four points of the quadrangle; determine the remaining three.

15. A system of circles passes through a fixed point, and cuts a given straight line in involution; prove that the circles all pass through a second fixed point.

16. Given a circle and a point; find the radical axis of the system of circles to which the given circle belongs, and of which the point is a limiting point.

17. Two parabolas can be drawn through four points on a circle; shew that they have their axes at right angles to one another, and parallel to the bisectors of the angles contained by any pair of chords through the four points.

18. A conic cuts the sides of a triangle in the six points $a_1 a_2, b_1 b_2, c_1 c_2$. All of these but one being given, shew how to determine that one with the ruler only.

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SOLUTIONS.

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1866.

I.

Algebra.

1. A man borrows P£ at r per cent. compound interest, and invests the sum in the r' per cent. stocks, the half-yearly dividends in which he immediately invests in the same stock. After the end of the fourth year, the stock rises in value 2 per cent. He then sells out, and, after repaying the loan with its accumulated interest, has Q£ left. But if the stock had not risen in value, he would have only just enough to repay the loan with its interest. Given P, Q, r, find the cost and percentage of the stocks in which he invested.

Let x = cost of stocks,

$P\left\{1+\frac{r}{100}\right\}^4$ = amount of loan with int. after 4 years,

$P.\frac{100}{x}\left\{1+\frac{r'}{200}\right\}^8$ = ... stock ....;

$\therefore$ by prob. $P\left\{1+\frac{r}{100}\right\}^4 = P.\left\{1+\frac{r'}{200}\right\}^8$,

$$P\left\{1+\frac{r}{100}\right\}^4 + Q = P.\frac{x+2}{x}\left\{1+\frac{r'}{200}\right\}^8;$$

whence $\left\{1+\frac{r}{100}\right\}^{\frac{1}{2}} = 1+\frac{r'}{200}$,

or $r' = 200\left\{\sqrt{1+\frac{r}{100}}-1\right\}$;

and $x = \frac{2P}{Q}\left\{1+\frac{r}{100}\right\}^4$.

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2. Eliminate x, y, z from

$$\frac{\frac{y}{z}+\frac{a}{x}}{y-z} = \frac{\frac{z}{x}+\frac{b}{y}}{z-x} = \frac{\frac{x}{y}+\frac{c}{z}}{x-y} \ (=r) \quad \ldots\ldots\ldots\ldots(1),$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab} \quad \ldots\ldots\ldots\ldots(2),$$

$$\frac{bc}{z^2}+\frac{ca}{x^2}+\frac{ab}{y^2} = \frac{z^2}{bc}+\frac{x^2}{ca}+\frac{y^2}{ab} \quad \ldots\ldots\ldots\ldots(3).$$

From (1)

$$r = \frac{\frac{xy}{z}+a+\frac{yz}{x}+b+\frac{zx}{y}+c}{xy-xz+yz-yx+zx-zy} = \frac{\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}+a+b+c}{0},$$

$$\therefore \quad x^2y^2+y^2z^2+z^2x^2 = -xyz\,(a+b+c);$$

from (2) $\quad x^2y^2+y^2z^2+z^2x^2 = \frac{x^2y^2z^2}{abc}(a+b+c),$

$$\therefore \quad xyz = -abc. \qquad (4)$$

Again, from (1),

$$r^3 = \frac{\left(\frac{y}{z}+\frac{a}{x}\right)\left(\frac{z}{x}+\frac{b}{y}\right)\left(\frac{x}{y}+\frac{c}{z}\right)}{(y-z)\,(z-x)\,(x-y)}$$

$$= \frac{1+\frac{abc}{xyz}+\frac{ac}{x^2}+\frac{ba}{y^2}+\frac{cb}{z^2}+\frac{bx}{yz}+\frac{cy}{zx}+\frac{az}{xy}}{(y-z)\,(z-x)\,(x-y)}$$

$$= \frac{1+\frac{abc}{xyz}+\frac{x^2}{ac}+\frac{bx}{yz}+\frac{y^2}{ba}+\frac{cy}{zx}+\frac{z^2}{cb}+\frac{az}{xy}}{(y-z)\,(z-x)\,(x-y)} \text{ from (3)}$$

$$= \frac{0}{(y-z)\,(z-x)\,(x-y)} \text{ from (4)};$$

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$\therefore$ one of the factors $y-z,\ z-x,\ x-y=0$;

put $y-z=0$;

$\therefore$ from (1), $\frac{y}{z}+\frac{a}{x}=0$, and $x=-a$;

$\therefore$ from (4), $y^2=z^2=bc$;

$\therefore$ from (2), $\frac{1}{a^2}+\frac{1}{bc}=\frac{1}{ca}+\frac{1}{ab}$;

or $a^2+bc=a(b+c)$.

3. Solve the equations:

(1) $x(x-1)(x-2)-a(a-1)(a-2)=0$;

(2)
$$\begin{cases} \frac{2ab}{x}+\frac{y}{a^2-b^2}=\frac{2ab}{a^2-b^2} \quad \ldots\ldots\ldots\ldots(\alpha), \\ \left(\frac{x}{a^2-b^2}\right)^2+\left(\frac{2ab}{y}\right)^2=\left(\frac{2ab}{a^2-b^2}\right)^2 \quad \ldots\ldots\ldots\ldots(\beta); \end{cases}$$

(3)
$$\begin{cases} (x+y)^{\frac{1}{2}}=-\frac{32}{65}xy \quad \ldots\ldots\ldots\ldots(\gamma), \\ \frac{65}{54}\left(1-\frac{y}{x}\right)^{\frac{1}{2}}=\frac{y}{y^{\frac{1}{2}}-x^{\frac{1}{2}}}-\frac{y}{y^{\frac{1}{2}}+x^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots(\delta). \end{cases}$$

(1) a is clearly a root,

$\therefore$ dividing by $x-a$,

$x^2+(a-3)x+a^2-3a+2=0$;

whence $x=\frac{3-a\pm\sqrt{1+6a-3a^2}}{2}$.

(2) from (α), $\frac{x}{a^2-b^2}+\frac{2ab}{y}=\frac{2ab}{a^2-b^2}\cdot\frac{x}{y}$;

$\therefore$ $\left(\frac{x}{a^2-b^2}\right)^2+\frac{4ab}{a^2-b^2}\cdot\frac{x}{y}+\left(\frac{2ab}{y}\right)^2=\left(\frac{2ab}{a^2-b^2}\right)^2\cdot\frac{x^2}{y^2}$;

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$$\therefore \text{ from } (\beta) \quad -\frac{4ab}{a^2-b^2}\cdot\frac{x}{y}=\left(\frac{2ab}{a^2-b^2}\right)^2-\left(\frac{2ab}{a^2-b^2}\right)^2\cdot\frac{x^2}{y^2};$$

$$\therefore \quad \frac{x}{y}=\frac{a^2-b^2\pm(a^2+b^2)}{2ab}=\frac{a}{b} \text{ or } -\frac{b}{a};$$

whence x and y are found.

(3) from (δ), $\dfrac{65}{54}\cdot\dfrac{(x-y)^{\frac{1}{2}}}{x^{\frac{1}{2}}}=\dfrac{2x^{\frac{1}{2}}y}{y-x}$;

$$\therefore \quad 65\,(x-y)^{\frac{3}{2}}=-108xy;$$

from (γ), $65\,(x+y)^{\frac{3}{2}}=-32xy$;

$$\therefore \quad \left(\frac{x-y}{x+y}\right)^{\frac{3}{2}}=\frac{27}{8};$$

$$\therefore \quad \frac{x-y}{x+y}=\frac{9}{4} \text{ or } \frac{9}{4}\alpha \text{ or } \frac{9}{4}\alpha^2,$$

where $1, \alpha, \alpha^2$ are the 3 values of $\sqrt[3]{1}$;

if we take the first value, we find $\dfrac{x}{y}=-\dfrac{13}{5}$,

whence from (γ) or (δ), $\quad x=\dfrac{13}{2}\,;\quad y=-\dfrac{5}{2}$.

4. If a series of terms in arithmetical progression is collected into groups of n terms, then the sums of these successive groups will form a series in arithmetical progression whose common difference is n^2 the original common difference.

Let a be the first term of any group, b the common difference of the series

$$S_1=a+\overline{a+b}+\ldots+\overline{a+n-1}.b=na+\frac{n(n-1)b}{2};$$

$$S_2=a+nb+\overline{a+n+1}.b+\ldots+\overline{a+2n-1}.b=n(a+nb)+\frac{n(n-1)b}{2};$$

$$\therefore \quad S_2-S_1=n^2b.$$

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5. Sum the series

$$2+4x+7x^2+12x^3+21x^4+38x^5+\ldots \text{ to } n \text{ terms,}$$

and find the value which x must not exceed in order that the nth term may be greater than all the following terms to infinity.

$$S_\infty = 1+2x+3x^2+4x^3+\ldots+1+2x+(2x)^2+(2x)^3+\ldots$$

$$= \frac{1}{(1-x)^2}+\frac{1}{1-2x};$$

$$S_n = \frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}+\frac{1-(2x)^n}{1-2x};$$

$$\therefore\ S_\infty - S_n = \frac{x^n}{1-x}\left\{n+\frac{1}{1-x}\right\}+\frac{(2x)^n}{1-2x};$$

and nth term $= nx^{n-1}+(2x)^{n-1}$;

$$nx^{n-1}+(2x)^{n-1} > \frac{x^n}{1-x}\left\{n+\frac{1}{1-x}\right\}+\frac{(2x)^n}{1-2x},$$

if $\quad 2n+2^n > \frac{n}{1-x}+\frac{2^{n-1}}{1-2x}+\frac{x}{(1-x)^2}$,

dividing by x^{n-1}, and adding $n+2^{n-1}$ to both sides:

this will always be the case if

$$2 \ll \frac{1}{1-2x},$$

i.e. if x is $\gtrsim \frac{1}{4}$.

7. . . . . What is the best number to back in throwing four dice; and on what other number are the odds the same as on eight?

The coefficient of x^r in $(x^1+x^2+x^3+x^4+x^5+x^6)^4$ gives the number of ways in which the number r can be thrown with 4 dice,

the greatest coefficient is that of x^{14},

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$\therefore$ 14 is the best number to back.

Also the coefficient of x^{20} is the same as that of x^8,

$\therefore$ 20 is equally likely to be thrown with 8.

NOTE. We may obtain a confirmation of the above result by taking the average throw;

the sum of the dots on the 6 faces of a die is 21,

$\therefore$ the average number is $3\frac{1}{2}$;

$\therefore$ the average throw for 4 dice will be 14.

8. There are ten counters in a bag marked with numbers. A person is allowed to draw two of them. If the sum of the numbers on these two is even, he receives as many shillings; if odd, he pays as many. Is the value of his expectation greater when the counters are numbered from 0 to 9 or from 1 to 10?

$$\text{Chance of drawing any particular two} = \frac{1}{\frac{10.9}{1.2.}} = \frac{1.2}{10.9};$$

$\therefore$ value of receiving expectation with counters marked from 0 to 9

$$= \frac{1.2}{10.9}\{\overline{0+2}+\overline{0+4}+\overline{0+6}+\overline{0+8}+\overline{1+3}+\overline{1+5}+\overline{1+7}$$
$$+\overline{1+9}+\overline{2+4}+\overline{2+6}+\overline{2+8}+\overline{3+5}+\overline{3+7}+\overline{3+9}$$
$$+\overline{4+6}+\overline{4+8}+\overline{5+7}+\overline{5+9}+\overline{6+8}+\overline{7+9}\} = 4;$$

so value of losing expectation $= 5$;

$\therefore$ value of whole expectation $= -1$.

If the counters are marked from 1 to 10,

value of receiving expectation $= 4\frac{8}{9}$,

value of losing expectation $= 6\frac{1}{9}$;

$\therefore$ value of whole expectation $= -1\frac{2}{9}$;

$\therefore$ the value of his expectation is greater in the first case.

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12. Prove the following properties of numbers:

(1) All cubes are of the form $9m$ or $9m \pm 1$;

(2) $n(n+1)(n+2)(5n^2-1)$ is divisible by 24.

(1) All numbers are of the form $3n$ or $3n \pm 1$;

$$(3n)^3 = 9m,$$
$$(3n \pm 1)^3 = 9m' \pm 1.$$

(2) $n(n+1)(n+2)(5n^2-1) = n(n+1)(n+2)\{4n^2+(n+1)(n-1)\}$

$$= 4n^2.n(n+1)(n+2)+(n+1)(n-1)n(n+1)(n+2);$$

and $n(n+1)(n+2)$ is divisible by 1.2.3,

$(n-1)n(n+1)(n+2)$ is divisible by 1.2.3.4.

14. The equation $x - \sqrt{a^2+x^2} = -\dfrac{na^2}{\sqrt{a^2+x^2}}$ is satisfied either by $x = \infty$ or $x = -a\dfrac{n-1}{\sqrt{2n-1}}$. But if the equation is solved the latter root only appears. Explain this.

$$x^2(a^2+x^2) = (a^2+x^2-na^2)^2 = x^4+2(1-n)a^2x^2+(1-n)^2a^4,$$

or $\quad 0.x^4+(2n-1)a^2x^2-(1-n^2)^2a^4 = 0;$

$\therefore \quad x^2 = \infty$ is a root of the equation.

15. Sum the series

(1) $1.3.2^2+3.5.3^2+5.7.4^2+\ldots\ldots$ to n terms;

(2) $\dfrac{3.6}{5.8}+\dfrac{4.7}{12.15}+\dfrac{5.8}{21.24}+\ldots\ldots$ to n terms;

(3) $\dfrac{1}{2.3}+\dfrac{3.5}{3.4.3}+\dfrac{5.5^2}{4.5.3^2}+\dfrac{7.5^2}{5.6.3^2}+\ldots$ to n terms.

(1) $S = \Sigma(2n+1)(2n+3)(n+2)^2$

$$= \frac{1}{4}\Sigma(2n+1)(2n+3)\{(2n-1)(2n+5)+4(2n+5)+1\}$$

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$$= \frac{1}{4}\Sigma.(2n+5)^{(4)} + \Sigma(2n+5)^{(3)} + \frac{1}{4}\Sigma(2n+3)^{(2)}$$

$$= \frac{(2n+5)^{(5)}}{40} + \frac{(2n+5)^{(4)}}{8} + \frac{(2n+3)^{(3)}}{24} + C$$

$$= \frac{(2n+3)(2n+1)(2n-1)}{8}\left\{\frac{(2n+5)(2n-3)}{5} + 2n + 5 + \frac{1}{3}\right\} + C$$

$$= \frac{(2n+3)(2n+1)(2n-1)}{120}\left\{12n^2+42n+35\right\} + C;$$

put $n = 0$; $\therefore \quad C = \frac{7}{8}$.

(2) $S = \Sigma\dfrac{(n+3)(n+6)}{(n+1)(n+2)(n+4)(n+5)} = \Sigma\dfrac{(n+3)^2(n+6)}{(n+1)(n+2)(n+3)(n+4)(n+}$

$$= \Sigma\frac{(n+3)(n+4)(n+5)-(n+4)(n+5)+n+5+(n+1)(n+2)+3(n+1)+}{(n+1)(n+2)(n+3)(n+4)(n+5)}$$

$$= \Sigma.(n+1)^{(-2)} - \Sigma(n+1)^{(-3)} + \Sigma(n+1)^{(-4)}$$
$$+ \Sigma(n+3)^{(-3)} + 3\Sigma(n+2)^{(-4)} + 4\Sigma(n+1)^{(-}$$

$$= C - (n+1)^{(-1)} + \frac{1}{2}(n+1)^{(-2)} - \frac{1}{3}(n+1)^{(-3)}$$
$$- (n+1)^{(-4)} - \frac{1}{2}(n+3)^{(-3)} - (n+2)^{(-3)}$$

$$= C - \frac{1}{n+1} + \frac{1}{2(n+1)(n+2)} - \frac{1}{3(n+1)(n+2)(n+3)}$$
$$- \frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{2(n+3)(n+4)} - \frac{1}{(n+2)(n+3)(n+}$$

$$= C - \frac{3n^3+27n^2+76n+67}{3(n+1)(n+2)(n+3)(n+4)};$$

put $n = 0$, then $C = \frac{67}{72}$.

(3) $S = \Sigma\dfrac{2n+1}{(n+2)(n+3)}\cdot\left(\frac{5}{3}\right)^n$.

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$$\text{Now } \frac{2n+1}{(n+2)(n+3)}\cdot\left(\frac{5}{3}\right)^n = \frac{5(n+2)-3(n+3)}{(n+2)(n+3)}\cdot\left(\frac{5}{3}\right)^n$$

$$= 3\left\{\frac{1}{n+3}\cdot\left(\frac{5}{3}\right)^{n+1} - \frac{1}{n+2}\cdot\left(\frac{5}{3}\right)^n\right\}$$

$$= 3\Delta\frac{1}{n+2}\cdot\left(\frac{5}{3}\right)^n;$$

$$\therefore\quad S = \frac{3}{n+2}\cdot\left(\frac{5}{3}\right)^n + C;$$

put $n = 0$, then $C = -\frac{3}{2}$.

II.

Trigonometry and Theory of Equations.

4. If from any point P in a circle lines are drawn to the extremities A, B and to the point of contact C of a side of the circumscribing square, shew that

$$\frac{(1+\cot PCA)^2}{\cot PBA} = \frac{(1+\cot PCB)^2}{\cot PAB}.$$

Draw PM perpendicular to AB;

then $PM^2 + CM^2 = 2\,AC.PM$, AC being = radius of circle;

$$\therefore\quad \{PM+CM\}^2 = 2\,PM.MB,$$

$$\{PM-CM\}^2 = 2\,PM.AM;$$

$$\therefore\quad \frac{\left\{1+\frac{CM}{PM}\right\}^2}{\left\{1-\frac{CM}{PM}\right\}^2} = \frac{\frac{MB}{PM}}{\frac{AM}{PM}};$$

$$\text{or}\quad \frac{(1+\cot PCA)^2}{\cot PBA} = \frac{(1+\cot PCB)^2}{\cot PAB}.$$

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5. If $\cos(a+\theta) = \cos a \cos\phi - \cos b \sin a \sin\phi$, and if θ and ϕ are small, shew that $\theta = \phi\cos b + \frac{\phi^2}{2}\cot a (\sin b)^2$, nearly.

$$\cos a\left\{1-\frac{\theta^2}{2}\right\} - \theta.\sin a = \cos a\left\{1-\frac{\phi^2}{2}\right\} - \phi\cos b.\sin a;$$

$$\therefore\quad \theta^2 + 2\theta\tan a + \tan^2 a = \tan^2 a + 2\phi\cos b.\tan a + \phi^2;$$

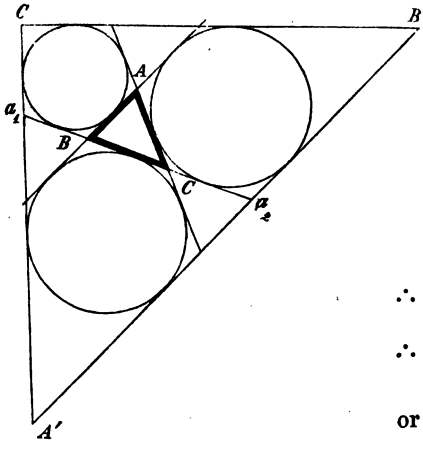
$$\therefore\quad \theta + \tan a = \tan a\left\{1+\left(\frac{2\phi\cos b}{\tan a} + \frac{\phi^2}{\tan^2 a}\right)\right\}^{\frac{1}{2}}$$

$$= \tan a\left\{1+\frac{1}{2}\left(\frac{2\phi\cos b}{\tan a} + \frac{\phi^2}{\tan^2 a}\right) - \frac{1}{8}\cdot\frac{4\phi^2\cos^2 b}{\tan^2 a}\right\} \text{ nearly};$$

$$\therefore\quad \theta = \phi\cos b + \frac{\phi^2}{2}\cot a (\sin b)^2 \text{ nearly}.$$

7. The triangle $A'B'C'$ circumscribes the escribed circles of the plane triangle ABC; shew that

$$\frac{B'C'}{a\cos A} = \frac{C'A'}{b\cos B} = \frac{A'B'}{c\cos C}.$$



Since AB, a_2A' are one pair of similar common tangents to the circles E_1, E_2, and BC, AC the other pair, it is clear that the angle

$$Ca_2A' = A;$$

so angle $Ba_1A' = A$;

$$\therefore\quad A' = \pi - 2A; \text{ \&c.}$$

$$\therefore\quad \frac{B'C'}{\sin 2A} = \frac{C'A'}{\sin 2B} = \frac{A'B'}{\sin 2C},$$

$$\text{or}\quad \frac{B'C'}{a\cos A} = \frac{C'A'}{b\cos B} = \frac{A'B'}{c\cos C}.$$

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8. A point is joined to the angular points of a polygon, whereby triangles are formed which have a common vertex, and have the sides of the polygon for their bases. For each triangle are drawn the inscribed circle, and the escribed circle which touches the base externally. Shew that the algebraic sum of the portions of the bases intercepted between the points of contact of these circles is zero.

Let O be the point, $A_1 A_2 \ldots$ the vertices of the polygon, then if I, E be the points of contact of the inscribed and escribed circles with the base of the triangle $OA_1 A_2$,

$$A_1 I = s - OA_2, \qquad A_1 E = s - OA_1,$$

where $2s$ = sum of sides;

$$\therefore \quad IE = OA_2 - OA_1;$$

$\therefore$ taking the triangles all round

$$\Sigma(IE) = OA_2 - OA_1 + OA_3 - OA_2 + \ldots\ldots = 0.$$

10. The angular points of a spherical triangle right-angled at C are joined by chords. If in the plane triangle thus formed C' corresponds to C, shew that

(1) $\cos C' = \sin \frac{a}{2} \sin \frac{b}{2}$;

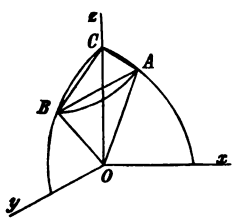
(2) $C' = \frac{\pi}{2} - \frac{ab}{4r^2}$, if the triangle is small.

(1) The angles which CA makes with the axes are

$$\frac{\pi}{2} - OCA, \quad \frac{\pi}{2}, \quad OCA,$$

or $$\frac{b}{2}, \quad \frac{\pi}{2}, \quad \frac{\pi}{2} - \frac{b}{2};$$

so the angles made by CB with the axes are $\frac{\pi}{2}, \quad \frac{a}{2}, \quad \frac{\pi}{2} - \frac{b}{2}$;



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therefore by means of the formula

$$\cos\theta = \cos a.\cos a' + \cos\beta.\cos\beta' + \cos\gamma.\cos\gamma'$$

we have $\cos C' = \sin\frac{a}{2}\sin\frac{b}{2}$.

(2) $\sin\frac{b}{2} = \frac{1}{2}\frac{CA}{r}$, $\sin\frac{a}{2} = \frac{1}{2}\frac{CB}{r}$;

if the triangle is small, $CA = b$, $CB = a$;

$$\therefore \quad \cos C' = \frac{ab}{4r^2};$$

$$\therefore \quad C' = \frac{\pi}{2} - \sin^{-1}\frac{ab}{4r^2}$$

$$= \frac{\pi}{2} - \frac{ab}{4r^2},$$

if a and b are small compared with r.

15. Sum the following series:

(1) $\frac{x\sin\theta}{1} + \frac{x^2\sin 2\theta}{1.2} + \frac{x^3\sin 3\theta}{1.2.3} + \ldots\ldots;$

(2) $(1-\sqrt{-1})-\sqrt{-2}-(1+\sqrt{-1})-\sqrt{2}-\ldots\ldots;$

and shew that 7 is the least number of terms the sum of which is rational.

(1) $S = \frac{x\sin\theta}{1} + \frac{x^2\sin 2\theta}{1.2} + \frac{x^3\sin 3\theta}{1.2.3} + \ldots\ldots,$

$$C = \frac{x\cos\theta}{1} + \frac{x^2\cos 2\theta}{1.2} + \frac{x^3\cos 3\theta}{1.2.3} + \ldots\ldots;$$

$$\therefore \quad C + S\sqrt{-1} = \frac{x(\cos\theta + \sqrt{-1}.\sin\theta)}{1} + \frac{x^2(\cos\theta+\sqrt{-1}.\sin\theta)^2}{1.2} + \ldots\ldots$$

$$= e^{x(\cos\theta+\sqrt{-1}\sin\theta)} - 1;$$

$$C - S\sqrt{-1} = e^{x(\cos\theta-\sqrt{-1}\sin\theta)} - 1;$$

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$$\therefore \quad S = \frac{e^{x(\cos\theta + \sqrt{-1}\sin\theta)} - e^{x(\cos\theta - \sqrt{-1}\sin\theta)}}{2\sqrt{-1}}$$
$$= e^{x\cos\theta}\sin(x\sin\theta).$$

(2) $S = (1-\sqrt{-1}) - \sqrt{-2} - (1+\sqrt{-1}) - \sqrt{2} - \ldots$ to n terms

$$= \sqrt{2}\left\{\left(\cos\frac{\pi}{4} - \sqrt{-1}\sin\frac{\pi}{4}\right) + \left(\cos\frac{\pi}{2} - \sqrt{-1}\sin\frac{\pi}{2}\right) + \ldots \right.$$
$$\left. \ldots + \left(\cos\frac{n\pi}{4} - \sqrt{-1}\sin\frac{n\pi}{4}\right)\right\}$$
$$= \frac{\sqrt{2}\left\{\cos\frac{\pi}{4} - \sqrt{-1}\sin\frac{\pi}{4}\right\}\left\{1 - \cos\frac{n\pi}{4} + \sqrt{-1}\sin\frac{n\pi}{4}\right\}}{1 - \cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}}$$
$$= \frac{\left\{\cos\frac{\pi}{4} - 1 - \sqrt{-1}\sin\frac{\pi}{4}\right\}\left\{1 - \cos\frac{n\pi}{4} + \sqrt{-1}\sin\frac{n\pi}{4}\right\}}{\sqrt{2}\left\{1 - \cos\frac{\pi}{4}\right\}}.$$

The coefficient of $\sqrt{-1}$ in the numerator

$$= \left\{\cos\frac{\pi}{4} - 1\right\}\sin\frac{n\pi}{4} - \left\{1 - \cos\frac{n\pi}{4}\right\}\sin\frac{\pi}{4}$$
$$= \sin\frac{\overline{n+1}.\pi}{4} - \sin\frac{n\pi}{4} - \sin\frac{\pi}{4} = 0 \text{ if } n = 7,$$

and vanishes for no less value of n.

III.

Pure Geometry.

2. If straight lines are drawn bisecting the interior angles of a quadrilateral, shew that they form another quadrilateral whose diagonals pass through the intersections of the opposite sides of the first.

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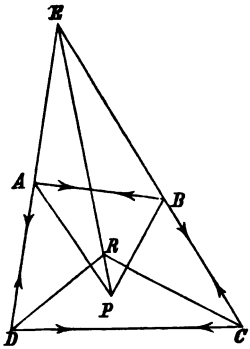
P is the centre of the escribed circle to ΔEAB; and R is the centre of the circle inscribed in ΔEDC;

$\therefore$ ER bisecting angle AEB plainly passes through P.

NOTE.—This problem affords a good illustration of a method of proving by mechanics that a number of points lie in a straight line.

Let eight equal forces be applied at A, B, C, D, two at each point, in directions $AB, AD, BA, BC, CB, CD, DC, DA$;



the resultant of the equal forces at A passes through P, as also that of the equal forces at B;

$\therefore$ P is a point on the resultant of the four equal forces; but since the equal forces along AB, BA neutralize each other,

$\therefore$ plainly the resultant of the four passes through E, i.e. EP is the line of the resultant.

Similarly ER is the line of the resultant of the four equal forces at C, D: but the resultant of the eight forces is zero, since they consist of four pairs equal and opposite;

$\therefore$ EP, ER must be the same straight line.

3. Given the vertex and the centres of the inscribed and circumscribed circles of a triangle, construct the triangle.

Take O, o, R, r, centres and radii of circumscribed and inscribed circles,

then $Oo^2 = R^2 - 2Rr$, whence r is known;

whence construct the inscribed circle, and draw tangents from the vertex; the points where these are cut by the circumscribed circle will be the other angular points of the triangle.

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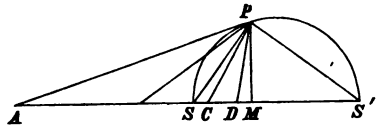
7. *A, B, C, D* **are fixed points in a given right line; find the locus of the point at which** *AB* **and** *CD* **subtend equal angles.**

Take S, S' the centres of similitude of the circles described on AB and CD as diameters, and on SS' as diameter describe a circle; this circle is the required locus; for since A, D and B, C are harmonic conjugates of S, S', and SPS' is a right angle,



$$\therefore \quad \angle APS = \angle SPD; \text{ and } \angle BPS = \angle SPC,$$
$$\therefore \quad \angle APB = \angle CPD.$$

This may also be proved analytically as follows:

Take $AB = a$; $BC = c$; $CD = b$; $AM = x$; $PM = y$; then $\tan(APM - BPM) = \tan(CPM - DPM)$;

$$\text{or} \quad \frac{\frac{x}{y} - \frac{x-a}{y}}{1 + \frac{x(x-a)}{y^2}} = \frac{\frac{x-a-c}{y} - \frac{x-a-b-c}{y}}{1 + \frac{(x-a-c)(x-a-b-c)}{y^2}};$$

$$\therefore \quad \frac{ay}{y^2 + x(x-a)} = \frac{by}{y^2 + (x-a-c)(x-a-b-c)};$$

$$\therefore \quad ay^2 + a(x-a-c)(x-a-b-c) = by^2 + bx(x-a),$$

$$\text{or} \quad x^2(a-b) + y^2(a-b) - 2a(a+c)x + a(a+c)(a+b+c) = 0.$$

8. A pair of ordinates *MP, M'P'* **perpendicular to the major axis of an ellipse intersect the auxiliary circle in** *Q, Q'*; **shew that, if** *C* **is the common centre of the curves, the areas of the triangles** *CQQ', CPP'* **are proportional to the major and minor axes of the ellipse. Hence shew that the area of the triangle formed by conjugate semi-diameters and the line joining their extremities is constant.**

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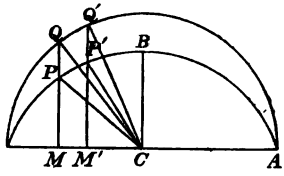
Every ordinate of the ellipse is to the corresponding ordinate of the auxiliary circle, as CB to CA;



$$\therefore \quad \text{quad}^l MQQ'M' : \text{quad}^l MPP'M' :: CA : CB,$$

and $\quad \Delta MQC : \Delta MPC :: CA : CB,$

and $\quad \Delta M'Q'C : \Delta m'P'C :: CA : CB;$

therefore

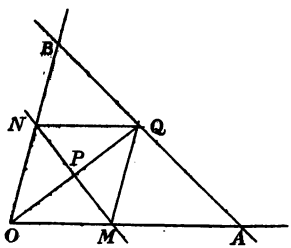
$$MQQ'M + M'Q'C - MQC : MPP'M + M'P'C - MPC :: CA : CB,$$

or $\quad \Delta CQQ' : \Delta CPP' :: CA : CB.$

When CP, CP' are conjugate diameters, CQ, CQ' are at right angles to each other;

$$\therefore \quad \Delta CPP' = \frac{CB}{CA} . \Delta CQQ' = \frac{1}{2} CA.CB.$$

9. Two sides of a parallelogram are fixed in position, and one vertex moves on a fixed straight line, shew that the envelope of one diagonal is a parabola.



The points at infinity on OA, OB are the centres of pencils of parallel lines; AB is their perspective axis: thus the pencils will determine homographic ranges on OA, OB; and since the pair of lines OA, OB may be regarded as a conic, lines joining corresponding points on the lines will envelope another conic touching AOB in A and B.

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Now take the position of the diagonal parallel to AB; then evidently P the middle point of the diagonal is a point on the locus; or MN is a tangent at P,

and $OP = PQ$;

but this is a property of the parabola.

14. Two triangles intersect in the same conic; shew that they are co-polar.

The sides of the triangles form an inscribed hexagon;

$\therefore$ by Pascal's theorem the points of intersection of opposite sides lie in a straight line,

i.e. the triangles are co-axal,

and co-axal triangles are also co-polar.

IV.

Analytical Geometry.

3. Shew that a straight line is represented by an equation of the first degree, when the coordinates of a point are the co-tangents of the angles which the lines joining the point to two fixed points make with the line joining the fixed points; interpret the equations

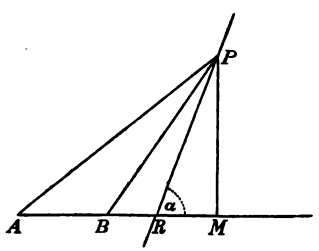
$$x+y=0, \qquad xy+1=0.$$

Take any line PR making an angle α with the line joining the fixed points: take any point P on it, and draw PM perpendicular to AB; then we have the following identical relation for the segments on the line AM:

$MA.BR + MB.RA + MR.AB = 0,$

or $BR.x + RA.y + AB \cot \alpha = 0,$

an equation of the first degree.



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$x+y=0$ represents the straight line perpendicular to AB through its middle point;

$xy+1=0$ represents the circle described on AB as diameter.

4. If p, q, r are the perpendiculars from the vertices of a triangle upon the line joining the centres of the inscribed and circumscribed circles, and a, b, c the sides of the triangle, prove that

$$\frac{ap}{-a+b+c}+\frac{bq}{a-b+c}+\frac{cr}{a+b-c}=0.$$

Take ABC as triangle of reference, and let O, I be the centres of circumscribed and inscribed circles.

Coordinates of A are $\frac{2\Delta}{a}, 0, 0,$

" O are $R\cos A, R\cos B, R\cos C,$

" I are $r \quad r \quad r$;

$\therefore$ area of triangle AOI is proportional to

$$\begin{vmatrix} \frac{1}{a} & 0 & 0 \\ \cos A & \cos B & \cos C \\ 1 & 1 & 1 \end{vmatrix}, \text{ i.e. to } \frac{\cos B-\cos C}{a},$$

or $ap=k(\cos B-\cos C)$;

so $bq=k(\cos C-\cos A)$, and $cr=k(\cos A-\cos B)$;

$$\cos B-\cos C=\frac{c^2+a^2-b^2}{2ca}-\frac{a^2+b^2-c^2}{2ab}=\frac{(c+a)^2-b^2}{2ca}-\frac{(a+b)^2-c^2}{2ab}$$

$$=\frac{(a+b+c)(c-b)(-a+b+c)}{2abc};$$

therefore

$$\frac{ap}{-a+b+c}+\frac{bq}{a-b+c}+\frac{cr}{a+b-c}=\frac{k(a+b+c)}{2abc}(c-b+a-c+b-a)=0.$$

6. Find the excentricity of an ellipse which is such that the vertex of an isosceles triangle whose base is the focal

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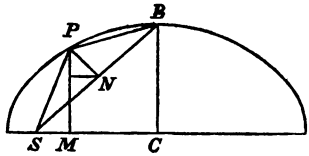
radius vector to the extremity of the minor axis, and vertical angle = 120°, will fall on the curve.

$$SN \sec PSN = \frac{a}{\sqrt{3}} = SP = a - eCM$$

$$= a - e\left\{\frac{a}{2}\cos BSC + PN.\sin BSC\right\} = a - \frac{ae^2}{2} - \frac{be}{2\sqrt{3}};$$



$$\therefore \quad 1 - \frac{1}{\sqrt{3}} - \frac{e^2}{2} - \frac{e}{2\sqrt{3}}\sqrt{1-e^2} = 0,$$

the equation for finding e.

7. Two straight lines including a right angle pass through the vertex of a triangle. Find the locus of intersection of one of them with the perpendicular drawn to the base at the point where it is cut by the other.

Take the given triangle as triangle of reference, and let C be the vertex;

let equation to one of the lines be $l\alpha - m\beta = 0$ ...(1);

then equation to the other will be

$$(l\cos C + m)\alpha + (l + m\cos C)\beta = 0 \text{.....(2)};$$

let the equation to the line perpendicular to $\gamma = 0$, at the point where it is met by (1), be $l'\alpha + m'\beta + n'\gamma = 0$;

since the three lines meet in a point,

we obtain $\quad l'm + lm' = 0$;

and since the line is perpendicular to $\gamma = 0$,

$$\therefore \quad l'\cos B + m'\cos A - n' = 0;$$

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$$\therefore \quad \frac{l'}{l} = \frac{-m'}{m} = \frac{n'}{l\cos B - m\cos A};$$

$\therefore$ the equation becomes $la - m\beta + (l\cos B - m\cos A)\gamma = 0$;

or $l(a + \cos B.\gamma) - m(\beta + \cos A.\gamma) = 0$;

from (2) $l(\cos C.a + \beta) + m(a + \cos C.\beta) = 0$;

$\therefore$ $(a + \cos B.\gamma)(a + \cos C.\beta) + (\beta + \cos A.\gamma)(\cos C.a + \beta) = 0$;

or $a^2 + \beta^2 + \sin B \sin C.\beta\gamma + \sin C \sin A.\gamma a + 2\cos C.a\beta = 0$.

10. A person wishing to ascertain the dip of a stratum bores from a horizontal plane vertical holes at the three corners of a square whose side is h; he finds the depths to be respectively a, b, c; determine the dip, and the depth of the stratum at the fourth corner of the square.

Take two sides of the square, and the vertical downwards through their intersection, as the axes of coordinates;

let the equation to the plane of the stratum be

$$Ax + By + Cz = D;$$

then we have $Ca = D$; $Ah + Cb = D$; $Bh + Cc = D$;

$$\therefore \quad \frac{A}{a-b} = \frac{B}{a-c} = \frac{C}{h} = \frac{D}{ah};$$

$\therefore$ equation to plane becomes $(a-b)x + (a-c)y + hz = ah$;

$\therefore$ the angle of the dip $= \cos^{-1} \dfrac{h}{\{(a-b)^2 + (a-c)^2 + h^2\}^{\frac{1}{2}}}$.

To find the depth at the fourth corner of the square, put $x = y = h$ in the equation of the plane,

$$\therefore \quad z = \frac{a}{2a - b - c}.$$

11. If

$$u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = xu_x + yu_y + u,$$

is the equation of a central conic, shew that the area con-

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tained by a chord whose middle point is (x', y') and lines drawn from its extremities to the centre is

$$\frac{\{\nabla u' - (ab-h^2)u'^2\}^{\frac{1}{2}}}{ab-h^2},$$

where ∇ is the eliminant of $u_x = 0$, $u_y = 0$, $u_1 = 0$.

$\frac{x-x'}{l} = \frac{y-y'}{m} = r$, the equation to a straight line,

to find where this meets the conic, substitute for x and y in the given equation,

$$\therefore (al^2 + 2hlm + bm^2)r^2 + 2\{lu'_x + mu'_y\}r + u' = 0 \ldots\ldots (1);$$

if (x', y') is the middle point of the chord

$$lu'_x + mu'_y = 0,$$

$$\text{or} \quad (x-x')u'_x + (y-y')u'_y = 0;$$

$$\therefore \quad x.u'_x + y.u'_y = x'.u'_x + y'.u'_y;$$

$$\therefore \quad x.u'_x + y.u'_y + u_1' = x'.u'_x + y'.u'_y + u_1' = u',$$

the equation to the chord in terms of its middle point.

For coordinates of centre we have $u_x = 0$, $u_y = 0$,

$$\text{or} \quad ax + hy + g = 0, \quad hx + by + f = 0;$$

$$\therefore \quad \frac{x}{A} = \frac{y}{B} = \frac{1}{C},$$

where A, B, C are the minors of the two top rows of

$$\nabla = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix};$$

$\therefore$ perpendicular from centre on chord

$$= \frac{Au'_x + Bu'_y + C(u_1' - u')}{C\{u'^2_x + u'^2_y\}^{\frac{1}{2}}}:$$

$$Au'_x + Bu'_y + Cu_1$$

$$= A(ax' + hy' + g) + B(hx' + by' + f) + C(gx' + fy' + c)$$

$$= (Aa + Bh + Cg)x' + (Ah + Bb + Cf)y' + Ag + Bf + Cc = \nabla,$$

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since the coefficients of x' and y' vanish, being determinants with two rows identical;

$$\therefore \text{ perpendicular} = \frac{\nabla - Cu'}{C\{u'_x{}^2 + u'_y{}^2\}^{\frac{1}{2}}} = \frac{\nabla - (ab-h^2)u'}{(ab-h^2)\{u'_x{}^2 + u'_y{}^2\}^{\frac{1}{2}}}.$$

Also from (1) length of $\frac{1}{2}$ chord $= \left\{\frac{-u'}{al^2 + 2hlm + bm^2}\right\}^{\frac{1}{2}}$,

where $lu'_x + mu'_y = 0$;

$$\therefore \frac{l}{u'_y} = \frac{-m}{u'_x} = \frac{1}{\{u'_x{}^2 + u'_y{}^2\}^{\frac{1}{2}}};$$

$$\therefore \{u'_x{}^2 + u'_y{}^2\}\{al^2 + 2hlm + bm^2\} = au'_y{}^2 - 2hu'_x u'_y + bu'_x{}^2$$

$$= a\{hx' + by' + f\}^2 - 2h(ax' + hy' + g)(hx' + by' + f) + b\{ax' + hy' + g\}^2$$

$$= \{ab - h^2\}\{ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c\} + g(bg - hf) + f(af - hg) + c(h^2 - ab)$$

$$= (ab-h^2)u' - \nabla;$$

$$\therefore \text{ area of triangle} = \tfrac{1}{2} \text{ perpendicular} \times \text{chord}$$

$$= \frac{\nabla - (ab-h^2)u'}{(ab-h^2)\{u'_x{}^2 + u'_y{}^2\}^{\frac{1}{2}}} \cdot \left\{\frac{u'\{u'_x{}^2 + u'_y{}^2\}}{\nabla - (ab-h^2)u'}\right\}^{\frac{1}{2}}$$

$$= \frac{\{\nabla u' - (ab-h^2)u'^2\}^{\frac{1}{2}}}{ab-h^2}.$$

12. (r_2, a_2) (r_3, a_3) are the focal coordinates of points on a conic section, ρ_1 the focal distance of the pole of the chord joining them; shew that, if $2b$ is the minor axis of the curve,

$$\frac{1}{r_2 r_3} - \frac{1}{\rho_1{}^2} = \frac{1}{b^2}\left(\sin\frac{a_2 - a_3}{2}\right)^2.$$

$\frac{l}{r} = 1 + e\cos\theta$, equation of the conic, where $l = a(1-e^2)$;

$\frac{l}{r} = e\cos\theta + \cos(\theta - a_2)$, equation of tangent at the point $(r_2 a_2)$;

$\frac{l}{r} = e\cos\theta + \cos(\theta - a_3)$, ... $(r_3 a_3)$;

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for point of intersection $\cos(\theta-\alpha_1)=\cos(\theta-\alpha_2)$;

$$\therefore\quad \theta=\frac{\alpha_1+\alpha_2}{2};$$

$$\therefore\quad \frac{l}{\rho_1}=e\cos\frac{\alpha_1+\alpha_2}{2}+\cos\frac{\alpha_1-\alpha_2}{2};$$

$$\therefore\quad \frac{l^2}{r_1r_2}-\frac{l^2}{\rho_1^2}$$

$$=(1+e\cos\alpha_1)(1+e\cos\alpha_2)-\left(e\cos\frac{\alpha_1+\alpha_2}{2}+\cos\frac{\alpha_1-\alpha_2}{2}\right)^2$$

$$=1-\cos^2\frac{\alpha_1-\alpha_2}{2}+e^2\left\{\cos\alpha_1\cos\alpha_2-\cos^2\frac{\alpha_1+\alpha_2}{2}\right\}$$

$$=1-\cos^2\frac{\alpha_1-\alpha_2}{2}-e^2\left\{1-\cos^2\frac{\alpha_1-\alpha_2}{2}\right\}=(1-e^2)\sin^2\frac{\alpha_1-\alpha_2}{2};$$

$$\therefore\quad \frac{1}{r_1r_2}-\frac{1}{\rho_1^2}=\frac{\sin^2\frac{\alpha_1-\alpha_2}{2}}{b^2}.$$

13. Shew that the point of intersection (x, y) of normals to the ellipse $b^2x^2+a^2y^2-a^2b^2=0$ at the extremities of the chord whose pole is (x', y') is determined by the equations

$$\frac{x}{x'(b^2-y'^2)}=\frac{-y}{y'(a^2-x'^2)}=\frac{a^2-b^2}{b^2x'^2+a^2y'^2}.$$

Find the coordinates of the centre of curvature for any point (x', y').

Equation of ellipse $\quad \frac{x^2}{a^2}+\frac{y_2}{b^2}=1$.....(1);

..... of polar of (x', y') $\quad \frac{xx'}{a^2}+\frac{yy'}{b^2}=1$.....(2);

..... of normal at (x_1, y_1) $\quad \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.....(3);

..... of normal at (x_2, y_2) $\quad \frac{a^2x}{x_2}-\frac{b^2y}{y_2}=a^2-b^2$.....(4);

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for intersection of (3) and (4),

$$\frac{a^2x}{\frac{1}{y_1}-\frac{1}{y_2}} = \frac{b^2y}{\frac{1}{x_1}-\frac{1}{x_2}} = \frac{a^2-b^2}{\frac{1}{x_2y_1}-\frac{1}{x_1y_2}},$$

$$\text{or} \quad \frac{a^2xy_1y_2}{y_2-y_1} = \frac{b^2yx_1x_2}{x_2-x_1} = \frac{(a^2-b^2)x_1x_2y_1y_2}{x_1y_2-x_2y_1} \quad \ldots\ldots(5).$$

Equation of line through $(x_1, y_1)(x_2, y_2)$ is

$$x(y_2-y_1)-y(x_2-x_1)-x_1y_2+x_2y_1 = 0;$$

this coincides with (2),

$$\therefore \quad \frac{a^2(y_2-y_1)}{x'} = \frac{-b^2(x_2-x_1)}{y'} = \frac{x_1y_2-x_2y_1}{1} \quad \ldots\ldots(6).$$

Again, to find the points where (2) meets (1),

$$\frac{a^2}{x'^2}\left\{1-\frac{yy'}{b^2}\right\}^2+\frac{y^2}{b^2} = 1;$$

$$\text{or} \quad \frac{y^2}{b^2}\left\{1+\frac{a^2y'^2}{b^2x'^2}\right\}-\frac{2a^2yy'}{b^2x'^2}+\frac{a^2}{x'^2}-1 = 0;$$

$$\left.\begin{aligned} \therefore \quad y_1y_2 &= \frac{b^4(a^2-x'^2)}{b^2x'^2+a^2y'^2}, \\ \text{so} \quad x_1x_2 &= \frac{a^4(b^2-y'^2)}{b^2x'^2+a^2y'^2}; \end{aligned}\right\} \quad \ldots\ldots(7)$$

$\therefore$ substituting in (5) from (6) and (7),

$$\frac{a^4b^4(a^2-x'^2)x}{\{b^2x'^2+a^2y'^2\}x'} = \frac{-a^4b^4(b^2-y'^2)y}{\{b^2x'^2+a^2y'^2\}y'} = \frac{(a^2-b^2)a^4b^4(a^2-x'^2)(b^2-y'^2)}{\{b^2x'^2+a^2y'^2\}^2};$$

$$\therefore \quad \frac{x}{x'(b^2-y'^2)} = \frac{-y}{y'(a^2-x'^2)} = \frac{a^2-b^2}{b^2x'^2+a^2y'^2}.$$

If $(x'y')$ is on the curve, it may be regarded as the pole of an infinitesimal chord; the above equations will then give the point of intersection of two consecutive normals, i.e. the centre of curvature at $(x'y')$.

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V.

Differential Calculus.

4. . . . Evaluate $\left(2-\frac{\sin x}{x}\right)^{(\cot x)^2}$, when $x=0$; and shew that

$$\left\{\frac{a_1^{nx}+a_2^{nx}+\ldots+a_n^{nx}}{n}\right\}^{\frac{1}{x}}=a_1 a_2 \ldots a_n, \text{ when } x=0.$$

(1) $u=\left\{2-\frac{\sin x}{x}\right\}^{(\cot x)^2}$;

$$\therefore \log u=\frac{\log\left\{2-\frac{\sin x}{x}\right\}}{(\tan x)^2}=\frac{\log\left\{2-\frac{1}{x}\left(x-\frac{x^3}{1.2.3}+\ldots\right)\right\}}{(\tan x)^2}$$

$$=\frac{\log\left\{1+\frac{x^2}{1.2.3}-\frac{x^4}{1.2.3.4.5}+\ldots\right\}}{\left\{x+\frac{x^3}{3}+\ldots\right\}^2}$$

$$=\frac{\frac{x^2}{1.2.3}-\frac{x^4}{1.2.3.4.5}-\frac{1}{2}\left\{\frac{x^2}{1.2.3}-\ldots\right\}^2}{x^2+\frac{2x^4}{3}+\ldots}=\frac{1}{6} \text{ when } x=0;$$

$$\therefore \quad u=e^{\frac{1}{6}}.$$

(2) $u=\left\{\frac{a_1^{nx}+a_2^{nx}+\ldots+a_n^{nx}}{n}\right\}^{\frac{1}{x}}$;

$$\therefore \log u=\frac{\log\{a_1^{nx}+a_2^{nx}+\ldots+a_n^{nx}\}-\log n}{x}$$

$$=\frac{a_1^{nx}\log a_1^n+a_2^{nx}\log a_2^n+\ldots+a_n^{nx}\log a_n^n}{a_1^{nx}+a_2^{nx}+\ldots+a_n^{nx}} \text{ when } x=0$$

$$=\log\{a_1 a_2 \ldots a_n\} \text{ when } x=0;$$

$$\therefore \quad u=a_1 a_2 \ldots a_n \text{ when } x=0.$$

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6. Find $f(x)$, when $f(x+y)+f(x-y) = 2xf(x^2-y^2)$.

Put $y = 0$; $\therefore\ f(x) = xf(x^2)$;

this suggests $f(x) = \frac{c}{x}$,

which is found to satisfy the equation.

Or thus—

expanding by Taylor's theorem, we have

$$f(x)+\frac{y^2}{1.2}f''(x)+\ldots = x\{f(x^2)-y^2f'(x^2)+\ldots\};$$

$\therefore$ equating coefficients of powers of y,

$$f(x) = xf(x^2) \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$f''(x) = -2xf'(x^2) \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$$\therefore\ f'(x) = f(x^2)+2x^2f'(x^2) = \frac{1}{x}f(x)-xf''(x);$$

$$\therefore\ x^2f''(x)+xf'(x)-f(x) = 0;$$

a differential equation of the form

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}-u = 0;$$

the solution of which is

$$u = f(x) = c_1x+\frac{c_2}{x};$$

and by trial in the original equation it is found that c_1 must $= 0$.

8. If $u = F(\xi, \eta, \zeta)$,

and $\xi = x^2-2yz$, $\eta = y^2-2zx$, $\zeta = z^2-2xy$, shew that

$$2(x^2+y^2+z^2+xyz)\left(\frac{du}{d\xi}\right) = \left(\frac{du}{dx}\right)(x^2-yz)-\left(\frac{du}{dy}\right)(z^2+xy)-\left(\frac{du}{dz}\right)(y^2+zx)$$

$$\frac{d\xi}{dx} = 2x; \quad \frac{d\xi}{dy} = -2z; \quad \frac{d\xi}{dz} = -2y;$$

&c. &c.

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therefore

$$\left(\frac{du}{dx}\right) = \left(\frac{du}{d\xi}\right)\cdot\frac{d\xi}{dx} + \left(\frac{du}{d\eta}\right)\frac{d\eta}{dx} + \left(\frac{du}{d\zeta}\right)\frac{d\zeta}{dx} = 2x\left(\frac{du}{d\xi}\right) - 2z\left(\frac{du}{d\eta}\right) - 2y\left(\frac{du}{d\zeta}\right);$$

so

$$\left(\frac{du}{dy}\right) = -2z\left(\frac{du}{d\xi}\right) + 2y\left(\frac{du}{d\eta}\right) - 2x\left(\frac{du}{d\zeta}\right);$$

$$\left(\frac{du}{dz}\right) = -2y\left(\frac{du}{d\xi}\right) - 2x\left(\frac{du}{d\eta}\right) + 2z\left(\frac{du}{d\zeta}\right);$$

$$\therefore \left(\frac{du}{dx}\right)(x^2-yz) - \left(\frac{du}{dy}\right)(z^2+xy) - \left(\frac{du}{dz}\right)(y^2+zx)$$

$$= 2(x^2+y^2+z^2+xyz)\left(\frac{du}{d\xi}\right).$$

13. . . . . **Find the locus of the extremities of lines drawn from a given point equal and parallel to the radii of curvature at the several points of an ellipse.**

Taking ξ, η as the current coordinates of the locus, x, y those of the ellipse, we have

$$\xi^2+\eta^2 = \frac{(a^4y^2+b^4x^2)^2}{a^8b^8},$$

$$\frac{\eta}{\xi} = \frac{a^2y}{b^2x}; \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \frac{\xi}{b^2x} = \frac{\eta}{a^2y} = \frac{\sqrt{\xi^2+\eta^2}}{\sqrt{b^4x^2+a^4y^2}} = \frac{\{\xi^2+\eta^2\}^{\frac{1}{2}}}{\{ab\}^{\frac{1}{2}}}$$

$$= \frac{\frac{\xi}{ab^2}}{\frac{x}{a}} = \frac{\frac{\eta}{a^2b}}{\frac{y}{b}} = \sqrt{\frac{\xi^2}{a^2b^4} + \frac{\eta^2}{a^4b^2}}$$

$$= \frac{1}{ab}\sqrt{\frac{\xi^2}{b^2} + \frac{\eta^2}{a^2}};$$

$$\therefore \left\{\frac{\xi^2+\eta^2}{ab}\right\}^{\frac{1}{2}} = \sqrt{\frac{\xi^2}{b^2} + \frac{\eta^2}{a^2}}.$$

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14. Find the curve in which the perpendicular from the origin on the tangent is one-half of the normal.

Perpendicular on tangent from origin

$$= \frac{y - x\frac{dy}{dx}}{\sqrt{1 + \frac{dy^2}{dx^2}}};$$

Normal $\quad = y\sqrt{1 + \frac{dy^2}{dx^2}};$

$$\therefore \quad y - x\frac{dy}{dx} = \frac{y}{2}\left(1 + \frac{dy^2}{dx^2}\right);$$

$$\therefore \quad \frac{dy^2}{dx^2} + \frac{2x}{y}\cdot\frac{dy}{dx} - 1 = 0;$$

$$\therefore \quad \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y};$$

$$\therefore \quad \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \pm\, dx;$$

$$\therefore \quad \sqrt{x^2+y^2} = c \pm x;$$

$$\therefore \quad y^2 = c^2 \pm 2cx;$$

the equation of a parabola.

15. Find the envelope of the osculating circle of a parabola.

Radius and coordinates of centre of curvature are

$$\frac{2(a+x)^{\frac{3}{2}}}{\sqrt{a}}, \quad 2a + 3x, \quad \frac{-xy}{a};$$

$\therefore$ equation of osculating circle is

$$\{\xi - 2a - 3x\}^2 + \left\{\eta + \frac{xy}{a}\right\}^2 = \frac{4(a+x)^3}{a} \quad \ldots\ldots\ldots\ldots (1);$$

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots (2);$$

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therefore

$$-6\{\xi-2a-3x\}dx+\frac{2}{a}\left\{\eta+\frac{xy}{a}\right\}\{x\,dy+y\,dx\}=\frac{12}{a}\{a+x\}^2dx,$$

$$y\,dy=2a\,dx;$$

$$\therefore\ -6y\{\xi-2a-3x\}+\frac{3y^2}{a}\left\{\eta+\frac{xy}{a}\right\}=\frac{12y}{a}\{a+x\}^2;$$

this reduces by means of (2) to

$$y^2-2\eta y+4a\xi=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

from (2) and (3) x and y are found, and substituting their values in (1),

we have the equation of the envelope.

VI.

Problems.

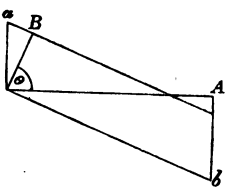
1. Two lines of lengths a and b are drawn from the same point and are inclined at an angle θ; from their extremities perpendiculars are drawn to each of them: find the area of the parallelogram thus formed.

$$Oa=OB\operatorname{cosec}\theta,$$

$$Ob=OA\operatorname{cosec}\theta;$$

$$\therefore\ \text{area}=Oa\,.\,Ob\,.\sin\theta$$

$$=OA\,.\,OB\operatorname{cosec}\theta.$$



2. Two numbers end with the same digit; and when they are divided by 7, the quotient of each is the remainder of the other: find the numbers.

The numbers necessarily consist of only two digits, let them be $a+10x$; $a+10y$;

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$$\left.\begin{aligned} a+10x &= 7p+q, \\ a+10y &= 7q+p; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\therefore \quad 10(x-y) = 6(p-q);$$

since x, y, p, q are all single digits, it is evident that

$$x-y = 3; \qquad p-q = 5 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$$\therefore \quad a+10x = 8p-5,$$

an indeterminate equation with three unknown quantities.

$\frac{a+5}{2} = 4p-5x$ is a whole number; $\quad \therefore \quad a$ is odd.

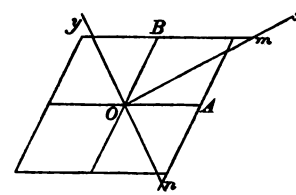
Put $a = 3$; then $x = 4$, $p = 6$, satisfy the equation, and also the conditions (2);

$\therefore$ 13, 43 are two such numbers as are required.

3. Ellipses are inscribed in a given parallelogram; shew that their foci lie on an equilateral hyperbola.

It is evident that the centre of the ellipse remains fixed, and coincides with the intersection of the diagonals of the parallelogram: for the self-conjugate triangle is the same for all conics inscribed in the same quadrilateral; and in the case of a parallelogram, one side of this triangle is at infinity, and its pole with respect to an inscribed conic is the intersection of diagonals. Take this as origin, and the lines bisecting the angle between the sides as axes.

Let $OA = a$, $OB = b$; $\angle BOA = 2\alpha$; $x', y', -x', -y'$, coordinates of foci;



equation of Bm,

$$y + x\tan\alpha - 2a\sin\alpha = 0,$$

equation of An,

$$y - x\tan\alpha + 2b\sin\alpha = 0;$$

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and since the product of perpendiculars from the foci on any tangent is constant,

$$\therefore \{y'+x'\tan a-2a\sin a\}\{-y'-x'\tan a-2a\sin a\}$$
$$=\{y'-x'\tan a+2b\sin a\}\{-y'+x'\tan a+2b\sin a\};$$

$$\text{whence} \quad x'y'=\frac{(a^2-b^2)\sin 2a}{2}.$$

Or thus—

Taking lines parallel to the sides as axes,

equation of ellipse is of form $Ax^2+Bxy+Cy^2=D$;

if the imaginary straight line $y=y'+\sqrt{-1}\,(x-x')$ touches the ellipse,

(x', y') being the only real point upon it must be a focus;

forming the condition for tangency in the usual way, we have after reduction

$$\{x'^2-y'^2\}\{4AC-B^2\}+4D(A-C)$$
$$-\{(B^2-4AC)x'y'-2BD\}\sqrt{-1}=0;$$

$$\therefore \{x'^2-y'^2\}\{4AC-B^2\}+4D(A-C)=0 \ldots\ldots\ldots (1);$$
$$(B^2-4AC)x'y'-2BD=0 \ldots\ldots\ldots (2);$$

two hyperbolae on which the foci lie.

Now since the lines $x=a$; $y=b$ are tangents to the ellipse,

$$\therefore \quad B^2a^2=4C(Aa^2-D); \qquad B^2b^2=4A(Cb^2-D);$$

$$\text{whence} \quad (a^2-b^2)(B^2-4AC)=4D(A-C);$$

$$\therefore \text{ from (1)} \qquad x'^2-y'^2=a^2-b^2.$$

4. The sum of three terms of a geometric series is given. Find, without using the differential calculus, what the terms of it must be in order that their continued product may be a maximum.

x, xr, xr^2 the terms;

then
$$x+xr+xr^2=c \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$x^3r^3=\mu \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

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from (2), $xr = \mu^{\frac{1}{3}}$, or $r = \dfrac{\mu^{\frac{1}{3}}}{x}$;

$\therefore$ from (1), $x + \mu^{\frac{1}{3}} + \dfrac{\mu^{\frac{1}{3}}}{x} = c$;

$$\therefore \quad x = \frac{-\mu^{\frac{1}{3}} + c \pm \sqrt{(\mu^{\frac{1}{3}} - c)^2 - 4\mu^{\frac{1}{3}}}}{2};$$

and since x must be rational, the greatest value of μ is obtained from $(\mu^{\frac{1}{3}} - c)^2 - 4\mu^{\frac{1}{3}} = 0$;

$$\therefore \quad \mu^{\frac{1}{3}} = xr = -c \text{ or } = \frac{c}{3};$$

$\therefore$ the terms are $c, -c, c$;

$$\text{or} \quad \frac{c}{3}, \frac{c}{3}, \frac{c}{3}.$$

5. If x is a number less than unity, then

$$\{(1-x)(1-x^3)(1-x^5)\ldots\ldots\}^{-1} = (1+x)(1+x^2)(1+x^3)\ldots\ldots,$$

the factors on each side being continued to infinity.

$$\begin{aligned}(1-x)(1+x)(1+x^2)(1-x^3)(1+x^3)(1+x^4)\ldots\ldots \\ &= (1-x^2)(1+x^2)(1+x^4)(1-x^6)\ldots\ldots \\ &= (1-x^4)(1+x^4)(1-x^8)\ldots\ldots \\ &= (1-x^8)(1-x^{16})\ldots\ldots \\ &= 1-x^{\infty} = 1 \text{ if } x<1.\end{aligned}$$

6. Shew that there is a point on the curve $y^2 = x^2 - x^3$, whose abscissa is $> \dfrac{9}{10}$ and < 1, such that the normal passes through the origin.

$$3y^2\,dy = (2x - 3x^2)\,dx;$$

equation to normal is $(\xi - x)\,dx + (\eta - y)\,dy = 0$;

$$\therefore \quad 3y^2(\xi - x) = (2x - 3x^2)(\eta - y).$$

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Put $\xi = \eta = 0$;

$$\therefore \quad y = \frac{2}{3} - x;$$

$\therefore$ from the equation to the curve

$$27x^3 - 36x + 8 = 0,$$

and a root of this lies between 1 and $\frac{9}{10}$.

7. In carrying out completely the process of finding the G. C. M. between two numbers which are prime to each other, the quotients in order are 5 three times and then 3 twice. Find the numbers.

Let a, b be the numbers, and c, d, e, f the other four divisors that occur in the process of finding the G. C. M.; then

$$3f = e; \quad 3e + f = d; \quad 5d + e = c; \quad 5c + d = b; \quad 5b + c = a;$$

whence $\quad 1428f = a; \quad 275f = b;$

$\therefore$ since a and b are prime to each other,

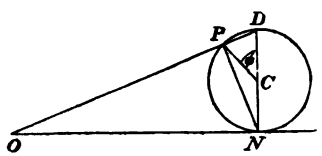
$$a = 1428; \quad b = 275.$$

8. From the origin of coordinates tangents are drawn to the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$. Shew that the coordinates of the point of contact of the tangent touching the nth branch are found by substituting for θ the value given by $\{(2n-1)\pi - \theta\} \tan\frac{\theta}{2} = 2$.

Let OP be the tangent: then OP is perpendicular to PN, and therefore passes through D the highest point of the circle.

Let $\angle DCP = \phi$;

$$\therefore \quad \angle DON = \frac{\phi}{2};$$



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$$\therefore \quad \tan\frac{\phi}{2} = \frac{DN}{ON} = \frac{2a}{2\pi a(n-1)+a(\pi-\phi)} = \frac{2}{(2n-1)\pi-\phi};$$

$$\therefore \quad \{(2n-1)\pi-\theta\}\tan\frac{\theta}{2} = 2$$

is the equation for determining the value of θ at the point P.

9. Find a fraction x and a whole number y such that $xy-\frac{y}{x}$ shall be the square of a whole number.

$$xy-\frac{y}{x} = p^2;$$

$$\therefore \quad x^2-\frac{p^2}{y}x-1 = 0;$$

$$\therefore \quad x = \frac{p^2+\sqrt{p^4+4y^2}}{2y}.$$

Find p^2 and $2y$ such that the quantity under the root shall be a perfect square;

then $$p^2 = l^2-m^2,$$

$$2y = 2lm;$$

$$\therefore \quad x = \frac{l^2-m^2+l^2+m^2}{2lm} = \frac{l}{m};$$

but since $$l^2-m^2 = p^2,$$

$$\therefore \quad l = s^2+t^2; \qquad m = 2st,$$

$$\therefore \quad x = \frac{s^2+t^2}{2st}; \qquad y = 2st(s^2+t^2).$$

10. Into a vessel containing a given quantity of a fluid A, another fluid B flows at a uniform rate; the fluids being at once completely mixed flow out of the vessel at the same rate. Determine the time at which the vessel contains an nth part of the original quantity of the fluid A.

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Let V be the volume of the fluid in the vessel, which remains constant, and let x be the amount of A at any time t; then the amount of A lost in time dt

$$= \frac{x}{V} \cdot dV = dx; \text{ and } dV = kdt;$$

$$\therefore \quad dt = \frac{V}{k} \cdot \frac{dx}{x};$$

$$\therefore \quad t = \frac{V}{k} \int_{\frac{V}{n}}^{V} \frac{dx}{x} = \frac{V}{k} \log n.$$

11. If in a series of numbers $a_1 b_1 c_1, a_2 b_2 c_2, \ldots a_n b_n c_n$, any group $a_r b_r c_r$ is connected with the preceding group $a_{r-1} b_{r-1} c_{r-1}$ by the equations

$$a_r a_{r-1} = b_{r-1} c_{r-1}, \quad b_r b_{r-1} = c_{r-1} a_{r-1}, \quad c_r c_{r-1} = a_{r-1} b_{r-1};$$

shew that $a_n = a^{\frac{1-(-2)^n}{3}} (bc)^{\frac{1-(-2)^{n-1}}{3}}$.

$$a_1 = a_1^{\frac{1-(-2)^1}{3}} (b_1 c_1)^{\frac{1-(-2)^{1-1}}{3}};$$

$$a_2 a_1 = b_1 c_1;$$

$$\therefore \quad a_2 = a_1^{-1} b_1 c_1 = a_1^{\frac{1-(-2)^2}{3}} (b_1 c_1)^{\frac{1-(-2)^{2-1}}{3}},$$

. . . . . .

let $a_{n-1} = a_1^{\frac{1-(-2)^{n-1}}{3}} (b_1 c_1)^{\frac{1-(-2)^{n-2}}{3}}$;

$$\therefore \quad a_n = a_{n-1}^{-1} b_{n-1} c_{n-1} = a_{n-1}^{-2} \cdot a_{n-1} b_{n-1} c_{n-1} = a_{n-1}^{-2} a_1 b_1 c_1$$

by the original equations

$$= a_1^{1+\frac{(-2)-(-2)^n}{3}} (b_1 c_1)^{1+\frac{(-2)-(-2)^{n-1}}{3}} = a_1^{\frac{1-(-2)}{3}} (b_1 c_1)^{\frac{1-(-2)^{n-1}}{3}}.$$

12. Q is the centre of curvature of the point P of an ellipse. A perpendicular is drawn to QP through Q. The part of this perpendicular cut off by the diameter through P is one-third of the radius of curvature of the evolute at Q.

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Equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

$$\text{radius of curvature} = R = \frac{\{a^2 - e^2x^2\}^{\frac{3}{2}}}{ab};$$

$$\therefore \quad dR = \frac{3e^2x(a^2-e^2x^2)^{\frac{1}{2}}}{ab}\,dx = \frac{3ae^2xy(a^2-e^2x^2)^{\frac{1}{2}}}{b(a^4y^2+b^4x^2)^{\frac{1}{2}}}\,ds$$

$$= \frac{3e^2xy\,.\,ds}{b^2}.$$

Now since any normal to the ellipse is a tangent to the evolute, $d\psi$ the angle between two consecutive normals is the same for both; and, if σ be an arc of the evolute,

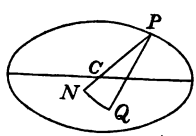
$$d\sigma = dR;$$

$\therefore$ ρ = radius of curvature of evolute

$$= \frac{dR}{d\psi} = \frac{3e^2xy}{b^2} \cdot \frac{ds}{d\psi} = \frac{3e^2xy}{b^2} \cdot R.$$

Again,

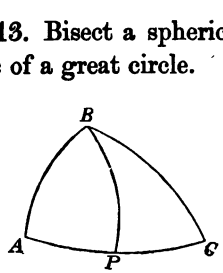
$$NQ = R\tan\{PGA - PCA\}$$



$$= \frac{\frac{a^2y}{b^2x} - \frac{y}{x}}{1 + \frac{a^2y^2}{b^2x^2}} \cdot R = \frac{e^2xy}{b^2}R;$$

$$\therefore \quad NQ = \frac{1}{3}\rho.$$

13. Bisect a spherical triangle having a right angle by an arc of a great circle.



$\angle APB = x$; $\quad \angle ABP = y$;

then we have

$$\frac{\pi}{2} + x + y - \pi = \frac{1}{2}\left\{\frac{\pi}{2} + B + C - \pi\right\};$$

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$$\text{or} \quad x+y=\frac{\pi}{4}+\frac{B+C}{2};$$

also, from the right-angled triangle BAP,

$$\cos x = \sin y \cos c;$$

whence x and y, and consequently AP, are found.

14. Find the equation of a circle which touches the lines $\alpha=0$, $\beta=0$, and the conic $\alpha\beta=k\gamma^2$.

Since $\alpha=0$, $\beta=0$ are tangents, the equation is of the form

$$\alpha\beta-(l\alpha+m\beta+n\gamma)^2=0 \ldots\ldots\ldots\ldots (1);$$

equation to a tangent at any point $(\alpha'\beta'\gamma')$ is

$$\{\beta'-2l(l\alpha'+m\beta'+n\gamma')\}\alpha+\{\alpha'-2m(l\alpha'+m\beta'+n\gamma')\}\beta$$
$$-2n(l\alpha'+m\beta'+n\gamma')\gamma=0; \ldots\ldots\ldots (2);$$

equation to a tangent to the given conic at the same point is

$$\beta'\alpha+\alpha'\beta-2k\gamma'\gamma=0 \ldots\ldots\ldots\ldots (3);$$

(2) and (3) must coincide,

$$\therefore \quad \frac{\beta'-2l(l\alpha'+m\beta'+n\gamma')}{\beta'}=\frac{\alpha'-2m(l\alpha'+m\beta'+n\gamma')}{\alpha'}$$
$$=\frac{n(l\alpha'+m\beta'+n\gamma')}{k\gamma'} \ldots\ldots\ldots (4);$$

$$\therefore \quad \frac{\beta'}{l}=\frac{\alpha'}{m}=\frac{\sqrt{k}.\gamma'}{\sqrt{lm}};$$

$\therefore$ substituting for $\alpha'\beta'\gamma'$ in the second and third of the equalities (4) we obtain

$$n=\sqrt{k}(1-2\sqrt{lm}) \ldots\ldots\ldots\ldots (5).$$

Again, applying to (1) the conditions that it should represent a circle, we have

$$(mc-nb)^2=(na-lc)^2=(lb-ma)^2+ab \ldots\ldots\ldots (6);$$

from (5) and (6) l, m, n are to be determined.

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15. Prove by the method of projections that of all polygons of a given number of sides circumscribing an ellipse, the one of least area is such that each side is parallel to the line joining the points of contact of adjacent sides.

We may use the following projective property, which is plainly true in the case of the circle:

If A_1, A_2, A_3 be the points of contact of three consecutive sides, the line joining the centre to the point of intersection of the tangents at A_1, A_3 passes through A_2.

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I.

Algebra.

1. If $b_1c_1+c_1a_1+a_1b_1=0$, and $a_2+b_2+c_2=0$, shew that

$$\frac{a_2b_1c_1}{b_1b_2-c_1c_2}+\frac{b_2c_1a_1}{c_1c_2-a_1a_2}+\frac{c_2a_1b_1}{a_1a_2-b_1b_2}=0;$$

and simplify

$$(a+b)\left\{a^2+b^2-ab+\frac{\sqrt{3}}{2}(a^2-b^2)\right\}^{\frac{1}{3}} \\ -(a-b)\left\{a^2+b^2+ab-\frac{\sqrt{3}}{2}(a^2-b^2)\right\}^{\frac{1}{3}}.$$

(1) $c_1=-\dfrac{a_1b_1}{a_1+b_1}$; $c_2=-(a_2+b_2)$;

$$\therefore \text{ expression} = \frac{-a_1a_2b_1^2}{b_1(b_1b_2-a_1a_2)}-\frac{a_1^2b_1b_2}{a_1(b_1b_2-a_1a_2)}-\frac{a_1b_1(a_2+b_2)}{a_1a_2-b_1b_2}=0.$$

(2) expression

$$=(a+b)\left\{\frac{1-\sqrt{3}}{2}a+\frac{1-\sqrt{3}}{2}b\right\}-(a-b)\left\{\frac{1-\sqrt{3}}{2}a-\frac{1+\sqrt{3}}{2}b\right\}$$

$$=\sqrt{3}.a^2+2ab-\sqrt{3}.b^2.$$

2. Solve the equations:

(1) $\sqrt[3]{2a+x}+\sqrt[3]{2a-x}=\sqrt[3]{a}$;

(2) $\left.\begin{aligned} x(x^2-3y^2)&=\frac{a^3}{\sqrt{2}},\\ x^2+y^2&=a^2;\end{aligned}\right\}$

(3) $2x^4+x^3+2x^2+3x+18=0$;

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(4) $a^2x+b^2y+c^2z=0$ .....(α),

$(b-c)^2yz+(c-a)^2zx+(a-b)^2xy=0$ .....(β),

$x^2+y^2+z^2=1$ .....(γ).

(1) $4a+3\sqrt[3]{4a^2-x^2}.\sqrt[3]{a}=a,$

$\therefore \quad 4a^2-x^2=-a^2,$

or $\quad x=\pm a\sqrt{5}.$

(2) Substituting for y^2

$$x^2-\frac{3a^2}{4}\cdot x-\frac{a^2}{4\sqrt{2}}=0;$$

or $$\left(x+\frac{a}{\sqrt{2}}\right)\left(x^2+\frac{a}{\sqrt{2}}\cdot x-\frac{a^2}{4}\right)=0.$$

(3) $$\left(x^2+\frac{x}{4}+3\right)^2=\frac{81x^2}{16}.$$

(4) From (β),

$$a^2(x^2+xy+xz)+b^2(yx+y^2+yz)+c^2(zx+zy+z^2)$$
$$=a^2x^2+b^2y^2+c^2z^2+2bcyz+2cazx+2abxy,$$

or $$(a^2x+b^2y+c^2z)(x+y+z)=(ax+by+cz)^2;$$

$\therefore$ from (α), $\quad ax+by+cz=0;$

and $\quad a^2x+b^2y+c^2z=0;$

$$\therefore \quad \frac{x}{bc(b-c)}=\frac{y}{ca(c-a)}=\frac{z}{ab(a-b)}$$

$$=\frac{1}{\{b^2c^2(b-c)^2+c^2a^2(c-a)^2+a^2b^2(a-b)^2\}^{\frac{1}{2}}} \text{ from } (\gamma).$$

3. Eliminate a and b from $ax=by=\frac{a^2b^2}{a^2+b^2}$, and

$$m=\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}.$$

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$$\frac{x}{b}=\frac{y}{a}=\left(\frac{xy}{ab}\right)^{\frac{1}{2}}=\left\{\frac{x^2+y^2}{a^2+b^2}\right\}^{\frac{1}{2}}=\frac{ab}{a^2+b^2}=\left(\frac{m^2}{ab}\right)^{\frac{1}{2}}=\left\{\frac{m^2}{a^2+b^2}\right\}^{\frac{1}{2}}$$

by given equations;

$$\therefore\quad a^2+b^2=\frac{(x^2+y^2)^2}{m^2};\quad ab=\frac{x^2y^2}{m^4};$$

$$\therefore\quad \left\{\frac{m^4}{x^2y^2}\right\}^{\frac{1}{2}}=\left\{\frac{m^4}{(x^2+y^2)^2}\right\}^{\frac{1}{2}};$$

$$\therefore\quad m^2(x^2+y^2)=x^2y^2.$$

5. The number $p_0+10^2p_1+10^4p_2+10^6p_3+\ldots+10^{2n}p_n$ is divisible by 37, if $p_0+p_1+\ldots+p_n$ be divisible by 37.

$$p_0+10^2p_1+\ldots+10^{2n}p_n=p_0+p_1+\ldots+p_n+(10^2-1)p_1$$
$$+(10^4-1)p_2+\ldots+(10^{2n}-1)p_n,$$

and $10^2-1=99$ is divisible by 37.

6. Shew that if a, b, c are unequal and $a+b+c>0$, then $a^3+b^3+c^3>3abc$; and that if x be real, then $\frac{x^2-x-5}{x-3}$ can have no real value between 3 and 7.

(1) $(b-c)^2+(c-a)^2+(a-b)^2>0$;

or $\quad a^2+b^2+c^2-bc-ca-ab>0$;

and $\quad a+b+c\qquad\qquad\qquad>0$;

$$\therefore\ (a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc>0.$$

(2) $\frac{x^2-x-5}{x-3}=y$;

$$\therefore\quad x=\frac{y+1\pm\sqrt{y^2-10y+21}}{2};$$

if x is real, $y^2-10y+21=(y-3)(y-7)>0$,

or y cannot lie between 3 and 7.

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7. Sum the following:

(1) $1+x(x+2)+x^2(x^2+2x+2)+x^3(x^3+2x^2+2x+2)+\ldots$ ad inf.

(2) $\frac{1+2}{1.2}+\frac{1+2+3}{1.2.3}+\frac{1+2+3+4}{1.2.3.4}+\ldots\ldots$ ad inf.

(3) $\frac{19}{1.2.3}\cdot\frac{2}{3}+\frac{28}{2.3.4}\cdot\left(\frac{2}{3}\right)^2+\frac{39}{3.4.5}\left(\frac{2}{3}\right)^3+\frac{52}{4.5.6}\cdot\left(\frac{2}{3}\right)^4+\ldots$ to n terms.

(1) $S=1+2x+3x^2+4x^3+5x^4+\ldots$

$$=\frac{1}{(1-x)^2}.$$

(2) $(r-1)$th term $=\frac{1+2+3\ldots+r}{1.2\ldots\ldots r}=\frac{r(r+1)}{2\lfloor r}=\frac{1}{2\lfloor r-2}+\frac{1}{\lfloor r-1}$;

$$\therefore\ S=\frac{1}{2}\left\{1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\right\}+\left\{\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\right\};$$

$$=\frac{e}{2}+e-1=\frac{3e}{2}-1.$$

(3) $(n+1)$th term $=\frac{19+n(n+8)}{(n+1)(n+2)(n+3)}\cdot\left(\frac{2}{3}\right)^{n+1}$

$$=\left\{\frac{6}{n+1}-\frac{7}{n+2}-\frac{2}{n+3}\right\}\left(\frac{2}{3}\right)^{n+1}$$

$$=3\left\{\frac{1}{n+3}\cdot\left(\frac{2}{3}\right)^{n+2}-\frac{1}{n+2}\cdot\left(\frac{2}{3}\right)^{n+1}\right\}-6\left\{\frac{1}{n+2}\cdot\left(\frac{2}{3}\right)^{n+2}-\frac{1}{n+1}\cdot\left(\frac{2}{3}\right)^{n+1}\right\}$$

$$=3\Delta\cdot\frac{1}{n+2}\cdot\left(\frac{2}{3}\right)^{n+1}-6\Delta\frac{1}{n+1}\cdot\left(\frac{2}{3}\right)^{n+1};$$

$$\therefore\ S=3\cdot\frac{1}{n+2}\cdot\left(\frac{2}{3}\right)^{n+1}-6\cdot\frac{1}{n+1}\cdot\left(\frac{2}{3}\right)^{n+1}+C$$

$$=-\frac{3(n+3)}{(n+1)(n+2)}\cdot\left(\frac{2}{3}\right)^{n+1}+C;$$

put $n=0$; $\qquad C=3.$

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9. **Three members are to be elected into the Hebdomadal Council, and no elector can vote for more than two candidates. What is the least proportion of the whole constituency which a minority can be in to be able to return one member, (1) when an elector may give a candidate one of his votes, but need not vote for more than one; (2) when an elector may give both his votes to the same candidate?**

Let N = whole number of voters,

x = the minority;

then, 1st, $x > \frac{2}{3}(N-x)$ or $> \frac{2N}{5}$;

2nd, $2x > \frac{2}{3}(N-x)$ or $x > \frac{N}{4}$.

10. **Shew that $3n$ is the most probable number to throw with $2n$ counters, each marked with 1 on one face and 2 on the other; also shew that the chance of throwing $3(n+1)$ with $2(n+1)$ counters is less than the chance of throwing $3n$ with $2n$ counters in the ratio $\frac{2n+1}{2n+2}$.**

(1) The coefficient of x^{3n} in $(x^1+x^2)^{2n}$

$=$ coefficient of x^n in $(1+x^2)^{2n}$,

and this is the largest coefficient.

(2) coefficient of $x^{3(n+1)}$ in $(x^1+x^2)^{2(n+1)}$

$=$ coefficient of x^{n+1} in $(1+x)^{3(n+1)} = P$;

coefficient of x^{3n} in $(x^1+x^2)^{2n}$

$=$ coefficient of x^n in $(1+x)^{2n} = Q$;

$$\frac{P}{Q} = \frac{2(2n+1)}{n+1};$$

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and no. of throws with $2(n+1)$ counters $= 2^{2(n+1)}$,

. . . . . $2n$ . . . . $= 2^{2n}$;

$$\therefore \quad \frac{\text{chance of } 3(n+1) \text{ with } 2(n+1)}{\text{chance of } 3n \text{ with } 2n} = \frac{2^{2n}}{2^{2(n+1)}} \cdot \frac{P}{Q} = \frac{2n+1}{2n+2}.$$

11. Find two fractions such that the quotients of their sum by their product and of their difference by their quotient shall be both squares.

$\frac{a}{b}, \frac{c}{d}$ the fractions,

$$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{ac}{bd}} = \frac{ac(ad+bc)}{a^2c^2} \text{ a square;}$$

$$\frac{\frac{a}{b}-\frac{c}{d}}{\frac{ad}{bc}} = \frac{ac(ad-bc)}{a^2d^2} \text{ a square;}$$

$$\therefore \quad ac(ad+bc) \text{ is a square} = mp \cdot \frac{p}{m},$$

$$ac(ad-bc) \quad . \quad . \quad . \quad = nq \cdot \frac{q}{n};$$

$$\therefore \quad ac = mp = nq; \quad ad+bc = \frac{p}{m}; \quad ad-bc = \frac{q}{n};$$

$$\therefore \quad \frac{a}{b}+\frac{c}{d} = \frac{p}{mbd}; \quad \frac{a}{b}-\frac{c}{d} = \frac{q}{nbd};$$

$$\therefore \quad \frac{4ac}{bd} = \frac{1}{b^2d^2}\left\{\frac{p^2}{m^2}-\frac{q^2}{n^2}\right\};$$

$$\therefore \quad \frac{1}{bd} = \frac{4m^2n^2p}{p^2n^2-q^2m^2} = \frac{4m^2n^4}{p(n^4-m^4)} \text{ since } mp = nq;$$

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$$\therefore \quad \frac{p}{mbd} = \frac{4m^2n^4}{n^4-m^4}; \quad \frac{q}{nbd} = \frac{4m^4n^2}{n^4-m^4};$$

$$\therefore \quad \frac{a}{b} = \frac{2m^2n^2}{n^2-m^2}; \quad \frac{c}{d} = \frac{2m^2n^2}{n^2+m^2}.$$

Or thus— x, y the fractions,

$$x+y = p^2xy,$$

$$x-y = q^2\frac{x}{y};$$

$$\therefore \quad x^2-y^2 = p^2q^2x^2;$$

$$\therefore \quad \frac{y^2}{x^2} = p^2\left(\frac{1}{p^2}-q^2\right) \text{ a square};$$

$$\therefore \quad \text{put} \quad p = \frac{1}{m}; \quad q = \frac{2m^2n}{n^2+m^2};$$

$$\therefore \quad \frac{1}{p^2}-q^2 = \frac{m^2(n^2-m^2)^2}{(n^2+m^2)^2};$$

$$\therefore \quad \frac{y}{x} = \frac{n^2-m^2}{n^2+m^2};$$

$$\therefore \quad x = \frac{2m^2n^2}{n^2-m^2}; \quad y = \frac{2m^2n^2}{n^2+m^2}.$$

II.

Trigonometry and Theory of Equations.

2. If A, B, C are the angles of a triangle,

$$\frac{\cos 3A}{\sin(A-B)\sin(A-C)} + \frac{\cos 3B}{\sin(B-C)\sin(B-A)} + \frac{\cos 3C}{\sin(C-A)\sin(C-B)} = 4.$$

$$\sin(B-C)\cos 3A = -\sin(A+2B)\cos 3A$$
$$= -\tfrac{1}{2}\sin 2(2A+B) - \tfrac{1}{2}\sin 2(A-B) = -\tfrac{1}{2}\sin 2(C-A) - \tfrac{1}{2}\sin 2(A-B);$$

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$$\therefore \quad \text{expression} = \frac{1}{\sin(B-C)\sin(C-A)\sin(A-B)}$$

$$\{\sin 2(C-A)+\sin 2(A-B)+\sin 2(B-C)\} = 4.$$

3. If R, r, r_a, r_b, r_c are the radii of the circumscribed, inscribed, and escribed circles respectively of a triangle, and C one of the angles, then $4R\cos C = r+r_a+r_b-r_c$.

$$r+r_a+r_b-r_c = S\left\{\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right\} = S\left\{\frac{-c}{s(s-c)}+\frac{c}{(s-a)(s-b)}\right\}$$

$$= \frac{c}{S}\left\{s^2-sc-s^2+s(a+b)-ab\right\} = \frac{c}{2S}(a^2+b^2-c^2)$$

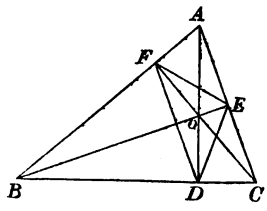
$$= \frac{2R}{ab}(a^2+b^2-c^2) = 4R\cos C.$$

4. If D, E, F are the points in the sides of the triangle ABC where the perpendiculars from the opposite angles meet the sides BC, CA, AB respectively, and if O is the point of intersection of these perpendiculars, then

$$\frac{AO}{\cos A} = \frac{BO}{\cos B} = \frac{CO}{\cos C} = 4R' = \frac{r'}{\cos A\cos B\cos C},$$

where R' and r' are the radii of the circumscribed and inscribed circles of the triangle DEF.

$$FE^2 = FA^2+AE^2-2FA.AE\cos A = (b^2+c^2-2bc\cos A)\cos^2 A = a^2\cos^2 A;$$



$$\therefore \quad FE = a\cos A;$$

$$DF = b\cos B;$$

$$ED = c\cos C;$$

also ΔAEF is similar to ΔABC &c.;

whence

$$\angle EDF = \pi - 2A \text{ &c.}$$

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$$\therefore \quad r' = \frac{1}{2} \cdot \frac{\text{area of } \Delta DEF}{\text{sum of sides}} = \frac{4\Delta ABC \cdot \cos A \cos B \cos C}{a\cos A + b\cos C + c\cos C},$$

$$\text{and } R' = \text{radius of nine-point circle} = \frac{R}{2};$$

therefore

$$4R' = 2R = \frac{OA}{\cos A} = \frac{OB}{\cos B} = \frac{OC}{\cos C} = \frac{a.OA + b.OB + c.OC}{a\cos A + b\cos B + c\cos C}$$

$$= \frac{a.AD + b.BE + c.CF - (a.OD + b.OE + c.OF)}{a\cos A + b\cos B + c\cos C}$$

$$= \frac{4\Delta ABC}{a\cos A + b\cos B + c\cos C} = \frac{r'}{\cos A \cos B \cos C}.$$

5. On the sides a, b of a right-angled triangle as diameters circles are described; shew that the sum of the distances of their common tangents from the right angle is

$$\frac{(a+b)ab}{a^2+b^2}.$$

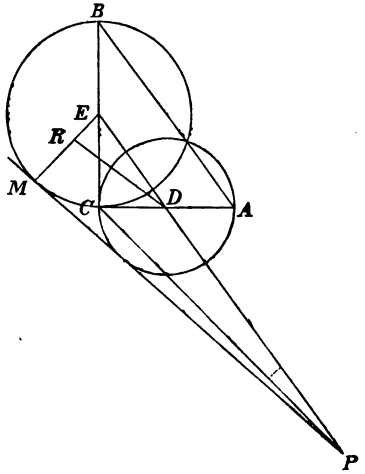
$$\frac{EP}{a} = \frac{DP}{b} = \frac{ED}{2ER}$$

$$= \frac{\sqrt{a^2+b^2}}{2(a-b)},$$

$$\frac{EP}{EC} = \frac{\sin ECP}{\sin\theta}$$

$$= \frac{\cos DCP}{\sin\theta},$$

$$\frac{DP}{DC} = \frac{\sin DCP}{\sin\theta};$$



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$$\therefore \quad \sin^2\theta\left\{\frac{EP^2}{EC^2}+\frac{DP^2}{DC^2}\right\}=1,$$

$$\text{or} \quad 2\sin^2\theta.\frac{a^2+b^2}{(a-b)^2}=1;$$

$$\therefore \quad \sin\theta=\frac{a-b}{\sqrt{2}.\sqrt{a^2+b^2}}; \qquad \cos\theta=\frac{a+b}{\sqrt{2}.\sqrt{a^2+b^2}};$$

$$CP=EC.\frac{\sin CED}{\sin\theta}=\frac{ab}{\sqrt{2}.(a-b)}.$$

$$\therefore \quad p_1+p_2=CP\{\sin(EPM+\theta)+\sin(EPM-\theta)\}$$

$$=2\,CP.\sin EPM\cos\theta=\frac{(a+b)\,ab}{a^2+b^2}.$$

6. The base BC of a triangle ABC is fixed, and the vertex A moves on the circumference of a circle which touches BC at its middle point A', and whose diameter is equal to BC; shew that

$$\sin 2\,AA'B=\frac{\sin(B-C)}{\sin(B+C)}.$$

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\frac{2\cos\frac{A}{2}\cos\frac{B-C}{2}}{b+c}=\frac{2\sin\frac{A}{2}\sin\frac{B-C}{2}}{b-c};$$

$$\therefore \quad \frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}.$$

$$b^2-c^2=CN^2-BN^2$$

$$=\left(\frac{a}{2}+\frac{a}{2}\sin 2\,AA'B\right)^2-\left(\frac{a}{2}-\frac{a}{2}\sin 2\,AA'B\right)^2$$

$$\{\text{since} \quad AOA'=2(\pi-APA')$$

$$=2(\pi-AA'B)\}$$

$$=a^2\sin 2\,AA';$$

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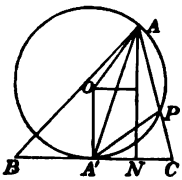
$$\therefore \quad \sin 2AA'B = \frac{b^2 - c^2}{a^2}$$

$$= \frac{\sin(B-C)}{\sin(B+C)}.$$



7. . . . . shew that

$$\sin^6\frac{\pi}{7} + \sin^6\frac{2\pi}{7} + \sin^6\frac{3\pi}{7} + \sin^6\frac{4\pi}{7} + \sin^6\frac{5\pi}{7} + \sin^6\frac{6\pi}{7} = 2\tfrac{3}{16}.$$

$$\text{expression} = 2\left\{\sin^6\frac{\pi}{7} + \sin^6\frac{2\pi}{7} + \sin^6\frac{3\pi}{7}\right\}$$

$$= \frac{1}{8}\left\{\left(3\sin\frac{\pi}{7} - \sin\frac{3\pi}{7}\right)^2 + \left(3\sin\frac{2\pi}{7} - \sin\frac{\pi}{7}\right)^2 + \left(3\sin\frac{3\pi}{7} - \sin\frac{2\pi}{7}\right)^2\right\}$$

$$= \frac{5}{4}\left\{\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{3\pi}{7}\right\} - \frac{3}{8}\left\{2\sin\frac{\pi}{7}\sin\frac{3\pi}{7} + 2\sin\frac{2\pi}{7}\sin\frac{\pi}{7} - 2\sin\frac{3\pi}{7}\sin\frac{2\pi}{7}\right\}$$

$$= \frac{5}{8}\left\{3 - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{\pi}{7}\right\} - \frac{3}{8}\left\{\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{2\pi}{7}\right\}$$

$$= \frac{15}{8} + \frac{5}{8}\left\{\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}\right\} = \frac{15}{8} + \frac{5}{8}\left\{\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}\right\}$$

$$= \frac{15}{8} + \frac{5}{16}\cdot\frac{\sin\frac{6\pi}{7}}{\sin\frac{\pi}{7}} \text{ by summation} = 2\tfrac{3}{16}.$$

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III.

Geometry.

5. The perpendiculars of a triangle ABC meet the opposite sides in A', B', C'; shew that the circles described about the triangles $AB'C'$, $BC'A'$, $CA'B'$ are together equal to 8 times the semicircles described on the perpendiculars from the centre of the circumscribing circle upon the sides.

$$AC' = b\cos A;\ AB' = c\cos A;\ C'B' = a\cos A;$$

$\therefore$ radius of circle circumscribing $AC'B' = \dfrac{C'B'}{2\sin A} = \dfrac{a}{2}\cot A;$

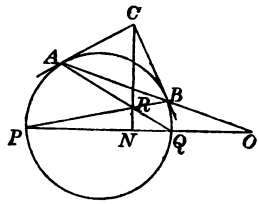
$\therefore$ sum of such circles $= \dfrac{\pi}{4}\{a^2\cot^2 A + b^2\cot^2 B + c^2\cot^2 C\}.$

Again, length of perpendicular from centre of circumscribing circle on $BC = \dfrac{a}{2}\cot A;$

$\therefore$ sum of semicircles on such perpendiculars

$$= \frac{\pi}{32}\{a^2\cot^2 A + b^2\cot^2 B + c^2\cot^2 C\}.$$

7. The locus of the intersections of lines drawn from the extremities of a variable diameter of a circle to two fixed points on the circle is a circle whose centre is the intersection of the tangents at the fixed points.



CR is the polar of O, and is therefore perpendicular to PQ, and PBQ is a right angle.

$\therefore$ $\angle BQP = \angle CRB$ the supplement of BRN.

also $\angle CBR = \angle BQP$ in the alternate segment;

$\therefore$ $\angle CBR = \angle CRB$;

$\therefore$ R lies on a circle, centre C, and radius CB.

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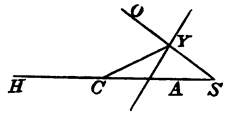
9. Given in position one focus and a tangent, and in length the transverse axis of a hyperbola; find the locus of the other focus.

Draw SY perpendicular to the tangent,



then Y is a fixed point, and $YC = AC$ is a fixed length;

$\therefore$ the locus of H is a circle whose centre is O (where $YO = SY$), and radius $= 2AC$.

11. Deduce a property of the parabola by reciprocating the following theorem: lines joining the extremities of a pair of diameters of a circle are parallel.

Take an extremity of one of the diameters as origin; then the circle reciprocates into a parabola; the two diameters, one into the point at infinity on the directrix, the other into any point on the directrix; the four points of intersection with the circle become the two pairs of tangents from the points on the directrix; and the line at infinity corresponds to the focus.

Hence. If a pair of tangents be drawn to a parabola from any point on the directrix, the line joining the intersection of one of them with the tangent at the vertex to the focus is parallel to the other.

(Strictly, the line joining the two points of intersection of the tangents passes through the focus; but one of these points lies at infinity.)

13. From any point P in a hyperbola, PN, Pn are drawn perpendicular to the axes, and PM, Pm to the asymptotes; prove that Nn and Mm are perpendicular to each other.

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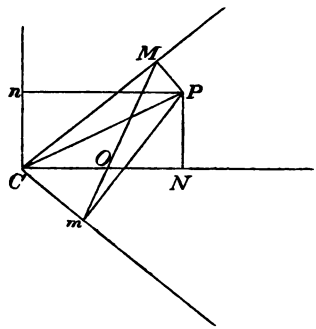
A circle can be described about the quadrilateral $MPmC$, which will also pass through N.



CN bisects the $\angle MCm = 2a$.

$$\angle MON = a + \angle CMm$$
$$= a + \angle CPm = \angle mPN + \angle CPm$$
$$= \angle CPN.$$

$\therefore$ Mm makes the same angle with axis of x, as Nn does with axis of y.

$\therefore$ they are perpendicular to each other.

IV.

Algebraical Geometry.

1. Shew that the area of the triangle contained by the lines

$$(1+bc)\,x+(1-c)\;y+(1+b)\;(2c)^{\frac{1}{2}} = 0 \;\ldots\ldots\; (1),$$
$$(1+c)\;x+(1+ca)y+(1-a)\;(2c)^{\frac{1}{2}} = 0 \;\ldots\ldots\; (2),$$
$$(1-b)\;x+(1+a)\;y+(1+ab)(2c)^{\frac{1}{2}} = 0 \;\ldots\ldots\; (3),$$

is $a+b+c+abc$.

for intersection of (1) and (2)

$$\frac{x}{(1-c)\,(1-a)-(1+ca)\,(1+b)} = \frac{-y}{(1+bc)\,(1-a)-(1+c)\,(1+b)}$$
$$= \frac{(2c)^{\frac{1}{2}}}{(1+bc)\,(1+ca)-(1+c)\,(1-c)};$$

or $$x = -y = \frac{-\sqrt{2c}}{c};$$

so for (1) and (3) $$x = \frac{y}{b} = -\sqrt{2c};$$

and for (2) and (3) $$\frac{x}{a} = -y = \sqrt{2c}.$$

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$$\therefore \quad \text{area} = \frac{1}{2}\begin{vmatrix} -\sqrt{\frac{2}{c}}, & \sqrt{\frac{2}{c}}, & 1 \\ -\sqrt{2c}, & -b\sqrt{2c}, & 1 \\ a\sqrt{2c}, & -\sqrt{2c}, & 1 \end{vmatrix} = a+b+c+abc.$$

4. Find the locus of the intersection of two tangents to an ellipse the product of the perpendiculars on which from the centre of the ellipse is equal to the product of the semi-axes.

If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, the product of perpendiculars from the origin $= \dfrac{c}{\{4h^2+(a-b)^2\}^{\frac{1}{2}}}$; (this is easily seen by taking the equations of two lines and multiplying them together.)

Now equation of pair of tangents to an ellipse from any point $(x'\ y')$ is

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right\}-\left\{\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right\}^2=0;$$

$\therefore$ product of perpendiculars from centre

$$= \frac{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}{\left\{\frac{4x'^2y'^2}{a^4b^4}+\left\{\frac{x'^2-y'^2-a^2+b^2}{a^2b^2}\right\}^2\right\}^{\frac{1}{2}}} = ab.$$

therefore

$$\frac{1}{a^2b^2}\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right\}^2 = \left\{\frac{x'^2+y'^2}{a^2b^2}\right\}^2 - \frac{2(x'^2-y'^2)(a^2-b^2)}{a^4b^4}+\frac{(a^2-b^2)^2}{a^4b^4};$$

$$\therefore \quad \frac{x'^4}{a^2}-\frac{y'^4}{b^2}-2(x'^2-y'^2)+a^2-b^2=0;$$

$$\text{or} \quad \left\{\frac{x^2}{a}+\frac{y^2}{b}-(a+b)\right\}\left\{\frac{x^2}{a}-\frac{y^2}{b}-(a-b)\right\}=0;$$

the equation of the locus.

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7. If $ax^2+2hxy+by^2+1=0$ represent a hyperbola, then its equation referred to its asymptotes, as coordinate axes, will be $4(h^2-ab)xy=\sqrt{(a-b)^2+4h^2}$.

Equations of asymptotes are $ax+(h-\sqrt{h^2-ab})y=0$;

$$ax+(h+\sqrt{h^2-ab})y=0.$$

Let θ be the angle between the asymptotes, x', y' the coordinates of any point on the curve referred to them; x, y its coordinates referred to the old axes, then

$$x'\sin\theta=\frac{ax+(h-\sqrt{h^2-ab})y}{\{a^2+2h^2-ab-2h\sqrt{h^2-a^2}\}^{\frac{1}{2}}};$$

$$y'\sin\theta=-\frac{ax+(h+\sqrt{h^2-ab})y}{\{a^2+2h^2-ab+2h\sqrt{h^2-ab}\}^{\frac{1}{2}}};$$

$$\therefore\quad x'y'\sin^2\theta=\frac{(h^2-ab)y^2-(ax+hy)^2}{\{(a^2+2h^2-ab)^2-4h^2(h^2-ab)\}^{\frac{1}{2}}}$$

$$=-\frac{a^2x^2+2ahxy+aby^2}{a\{(a-b)^2+4h^2\}^{\frac{1}{2}}}=\frac{1}{\{(a-b)^2+4h^2\}^{\frac{1}{2}}};$$

Also $$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b};$$

$$\therefore\quad \sin^2\theta=\frac{4(h^2-ab)}{(a-b)^2+4h^2};$$

$$\therefore\quad 4x'y'(h^2-ab)=\sqrt{(a-b)^2+4h^2}.$$

8. Normals are drawn to an equilateral hyperbola from a given point; shew that the sum of their squares is equal to three times the square of the distance of the given point from the centre of the hyperbola.

(a, b) the given point; N the length of a normal

$$\Sigma.N^2=4(a^2+b^2)-2a\Sigma x-2b\Sigma y+\Sigma x^2+\Sigma y^2.$$

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Equation of hyperbola $x^2-y^2=c^2$,

. . . . normal $\frac{a}{x}+\frac{b}{y}=2$,

where (x, y) is the unknown point of contact.

$$y=\frac{b}{2x-a};$$

$$\therefore\quad x^2(2x-a)^2-b^2x^2=c^2(2x-a)^2;$$

or $$4x^4-4ax^3+(a^2-b^2-4c^2)x^2+\ldots=0;$$

$$\therefore\quad \Sigma x=a;\qquad \Sigma xx'=\frac{a^2-b^2-4c^2}{4}.$$

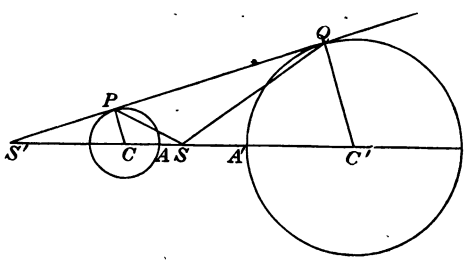
$$\therefore\quad \Sigma x^2=a^2-\frac{a^2-b^2-4c^2}{2}=\frac{a^2+b^2}{2}+2c^2.$$

similarly $$\Sigma y=b;\qquad \Sigma y^2=\frac{a^2+b^2}{2}-2c^2.$$

$$\therefore\quad \Sigma N^2=3(a^2+b^2).$$

9. Lines are drawn from the internal centre of similitude of two circles to the points of contact of an external common tangent; shew that if α, β, γ are the angles which these lines and the tangent make with the line joining the centres of the circles, then $\cot\alpha+\cot\beta+2\tan\gamma=0$.

$\angle QSC'=\alpha$;
$\angle PSC'=\beta$;
$\angle PS'C=\gamma$;
$CP=a$;
$C'Q=b$;
$AA'=c$.



$$\frac{SC}{a}=\frac{SC'}{b}=\frac{a+b+c}{a+b}$$

$$\frac{\sin QSC'}{\sin C'QS}=\frac{\sin\alpha}{\cos(\alpha-\gamma)}=\frac{a+b}{a+b+c};$$

$$\frac{\sin PSC}{\sin CPS}=\frac{-\sin\beta}{\cos(\gamma-\beta)}=\frac{a+b}{a+b+c};$$

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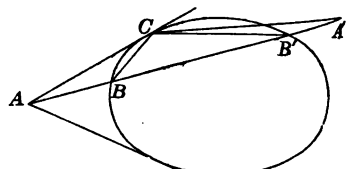
$$\therefore \quad \frac{\sin\alpha}{\cos(\alpha-\gamma)} + \frac{\sin\beta}{\cos(\gamma-\beta)} = 0;$$

whence $\quad \cot\alpha + \cot\beta + 2\tan\gamma = 0.$

10. If from a fixed point A chords are drawn to a conic intersecting it in B, B', and a fourth point A' is taken so that the anharmonic ratio of $ABB'A'$ is constant, shew that the locus of A' is a conic having double contact with the given conic.

Equation of conic referred to pair of tangents from A, and their chord of contact is $\quad LM = R^2.$



Equation of ABB' is

$$\mu^2 L = M.$$

$\therefore$ if B is the point $+\mu$, B' will be $-\mu$.

Thus equation of CA is $\quad L = 0;$

. . . . . CB . $\quad \mu L - R = 0;$

. . . . . CB' . $\quad \mu L + R = 0;$

. . . . . CA' . $\quad kL - R = 0.$

$$\text{anharmonic ratio} = \frac{\frac{1}{\mu}}{\frac{1}{k}} \div \frac{\frac{2}{\mu}}{\frac{1}{\mu}+\frac{1}{k}} = \frac{\mu+k}{2\mu} = \text{const.} = \lambda.$$

$$\therefore \quad k = \mu(2\lambda-1);$$

$\therefore$ equation of CA' is $\mu(2\lambda-1)L - R = 0,$

and . . . . AA' . $\mu^2 L - M = 0.$

$\therefore$ for point of intersection

$\mu^2(2\lambda-1)^2 L^2 = R^2 = (2\lambda-1)^2 LM$ (since $\mu^2 L = M$);

the equation of a conic having double contact with $R^2 = LM$.

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11. Normals at the extremity of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ intersect the polar of the focus and each other at points whose ordinates are y_1, y_2, y_3; y_1', y_2', y_3' are similar ordinates for a focal chord at right angles to the first; shew that

$$\left(\frac{y_1+y_2-2y_3}{y_1-y_2}\right)^2+\left(\frac{y_1'+y_2'-2y_3'}{y_1'-y_2'}\right)^2=\frac{a^2}{a^2-b^2}.$$

Equations of normals $\frac{a^2x}{x'}-\frac{b^2y}{y'}=a^2-b^2;$

$$\frac{a^2x}{x''}-\frac{b^2y}{y''}=a^2-b^2;$$

putting $x=\frac{a}{e}$, we have

$$y_1=\frac{y'}{b^2}\left(\frac{a^2}{ex'}-a^2+b^2\right); \qquad y_2=\frac{y''}{b^2}\left(\frac{a^2}{ex''}-a^2+b^2\right).$$

Now if $(x'\,y')$ $(x''\,y'')$ are the extremities of a focal chord, and θ the angle it makes with the axis of x

$$y'=(a-ex')\sin\theta; \qquad x'-ae=(a-ex')\cos\theta;$$

$$\therefore\quad x'=\frac{a(e+\cos\theta)}{1+e\cos\theta}; \qquad y'=\frac{b^2\sin\theta}{a(1+e\cos\theta}};$$

and $$x''=\frac{a(e-\cos\theta)}{1-e\cos\theta}; \qquad y''=\frac{-b^2\sin\theta}{a(1-e\cos\theta)};$$

$$\therefore\quad y_1 \text{ becomes} =\frac{b^2}{a}\cdot\frac{\sin\theta(1+e\cos\theta+e^2)}{e(1+e\cos\theta)(e+\cos\theta)};$$

$$y_2=\frac{-b^2}{a}\cdot\frac{\sin\theta(1-e\cos\theta+e^2)}{e(1-e\cos\theta)(e-\cos\theta)};$$

and $y_3=\frac{y'+y''}{2}$ (by a well-known property) $=\frac{-b^2}{a}\,\frac{e\sin\theta\cos\theta}{1-e^2\cos^2\theta};$

$$\therefore\quad \frac{y_1+y_2-2y_3}{y_1-y_2}=-\frac{\cos\theta}{e} \text{ after reduction;}$$

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$$\text{so} \quad \frac{y_1'+y_2'-2y_3'}{y_1'-y_2'} = \frac{\sin\theta}{e};$$

$$\therefore \quad \left\{\frac{y_1+y_2-2y_3}{y_1-y_2}\right\}^2 + \left\{\frac{y_1'+y_2'-2y_3'}{y_1'-y_2'}\right\}^2 = \frac{1}{e^2} = \frac{a^2}{a^2-b^2}.$$

12. . . . . . Equilateral hyperbolas are described about a given triangle; then the locus of their centres is the circle that passes through the middle points of the sides of the triangle.

Equation of circumscribed conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0. \quad \ldots\ldots\ldots\ldots \quad (1).$$

Condition for equilateral hyperbola

$$l\cos A + m\cos B + n\cos C = 0. \quad \ldots\ldots\ldots \quad (2).$$

Equations for centre of (1) are

$$\frac{m\gamma+n\beta}{a} = \frac{n\alpha+l\gamma}{b} = \frac{l\beta+m\alpha}{c};$$

or

$$\frac{l}{a\alpha^2-b\alpha\beta-c\gamma\alpha} = \frac{m}{b\beta^2-c\beta\gamma-a\alpha\beta} = \frac{n}{c\gamma^2-a\gamma\alpha-b\beta\gamma}$$

$$= \frac{l\cos A + m\cos B + n\cos C}{(a\alpha^2-b\alpha\beta-c\gamma\alpha)\cos A + \ldots\ldots};$$

$\therefore$ from (2) $(a\alpha^2-ba\beta-c\gamma\alpha)\cos A + \ldots\ldots = 0;$

or $a\beta\gamma + b\gamma\alpha + c\alpha\beta = a\alpha^2\cos A + b\beta^2\cos B + c\gamma^2\cos C;$

the equation of the nine point circle.

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V.

Differential Calculus.

4. Evaluate $\left(\frac{x}{a}\right)^{\frac{1}{\sqrt{a^2-x^2}}}$, when $x = a$; and shew that if $a+b+c = 0$ and $x = 0$, $\frac{\tan ax + \tan bx + \tan cx}{\tan ax \tan bx \tan cx} = 1$.

$$u = \left(\frac{x}{a}\right)^{\frac{1}{\sqrt{a^2+x^2}}};$$

$$\therefore \log u = \frac{\log \frac{x}{a}}{\sqrt{a^2-x^2}}$$

$$= -\frac{\sqrt{a^2-x^2}}{x^2} \text{ when } x = a$$

$$= 0 \quad \ldots\ldots$$

$$\therefore u = 1.$$

$$\tan(a+b+c)x = 0 = \frac{\tan ax + \tan bx + \tan cx - \tan ax \tan bx \tan cx}{1 - \tan ax \tan bx - \tan bx \tan cx - \tan cx \tan ax};$$

$$\therefore \tan ax + \tan bx + \tan cx - \tan ax \tan bx \tan cx = 0.$$

5. Two similar ellipses are such that the major axis of the one coincides with the minor axis of the other both in length and direction; shew that the maximum chord of the larger ellipse which touches the other is equal to the line joining the extremities of the axes of the larger ellipse.

Equation of larger ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ..... (1).

Equation of smaller ellipse $\frac{a^2x^2}{b^4} + \frac{y^2}{b^2} = 1$ ..... (2).

Equation of chord of (1) in terms of its middle point is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2};$$

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Equation of tangent to (2) at any point is

$$\frac{a^2xx'}{b^4} + \frac{yy'}{b^2} = 1;$$

if these coincide $\dfrac{\dfrac{x_1}{a^2}}{\dfrac{a^2x'}{b^4}} = \dfrac{y_1}{y'} = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}$;

or $$\frac{\dfrac{b^2x_1}{a^2}}{\dfrac{ax'}{b^2}} = \frac{\dfrac{y_1}{b}}{\dfrac{y'}{b}} = \left\{\frac{b^4x_1^2}{a^2} + \frac{y_1^2}{b^2}\right\}^{\frac{1}{2}} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \quad \ldots\ldots (3).$$

Now if we call $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = \mu^2$, $\qquad \dfrac{b^2x_1^2}{a^2} + \dfrac{a^2y_1^2}{b^2} = \beta^2$,

then $$\frac{\dfrac{x_1^2}{a^2}}{\beta^2 - a^2\mu^2} = \frac{-\dfrac{y_1^2}{b^2}}{\beta^2 - b^2\mu^2} = \frac{-1}{a^2+b^2} = \frac{\dfrac{b^4x_1^2}{a^4} + \dfrac{y_1^2}{b^2}}{\dfrac{b^4}{a^4}(\beta^2 - a^2\mu^2) - \beta^2 + b^2\mu^2}$$

$$= \frac{\mu^4}{\dfrac{b^4}{a^4}(\beta^2 - a^2\mu^2) - \beta^2 + b^2\mu^2} \text{ from (3);}$$

$$\therefore \quad a^4\mu^4 + a^2b^2\mu^2 - (a^2+b^2)\beta^2 = 0 \quad \ldots\ldots\ldots\ldots (4).$$

Also length of half chord in terms of its middle point

$$= \frac{\left\{\dfrac{b^2x_1^2}{a^2} + \dfrac{a^2y_1^2}{b^2}\right\}^{\frac{1}{2}} \left\{1 - \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2}\right\}^{\frac{1}{2}}}{\left\{\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}\right\}^{\frac{1}{2}}} = \frac{\beta}{\mu}\sqrt{1-\mu^2};$$

and putting $\mu^2 = x$; $\beta^2 = y$

we have to find $\dfrac{y}{x} - y$ a max., subject to the condition

$$a^4x^2 + a^2b^2x - (a^2+b^2)y = 0;$$

the value is easily found to be $\dfrac{a^2+b^2}{4}$;

thus length of chord $= \sqrt{a^2+b^2}$.

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6. If $u = f(l, m, n)$, and

$$l = \frac{y-z}{x}, \quad m = \frac{z-x}{y}, \quad n = \frac{x-y}{z}, \quad \text{then}$$

$$(1+mn)\left(\frac{du}{dx}\right)+(1-n)\left(\frac{du}{dy}\right)+(1+m)\left(\frac{du}{dz}\right) = 0.$$

$$\left(\frac{du}{dx}\right)=\left(\frac{du}{dl}\right)\frac{dl}{dx}+\left(\frac{du}{dm}\right)\left(\frac{dm}{dx}\right)+\left(\frac{du}{dn}\right)\frac{dn}{dx}=-\frac{l}{x}\left(\frac{du}{dl}\right)-\frac{1}{y}\left(\frac{du}{dm}\right)+\frac{1}{z}\left(\frac{du}{dn}\right);$$

$$\left(\frac{du}{dy}\right) = \frac{1}{x}\left(\frac{du}{dl}\right)-\frac{m}{y}\left(\frac{du}{dm}\right)-\frac{1}{z}\left(\frac{du}{dn}\right);$$

$$\left(\frac{du}{dz}\right) = -\frac{1}{x}\left(\frac{du}{dl}\right)-\frac{1}{y}\left(\frac{du}{dm}\right)-\frac{n}{z}\left(\frac{du}{dn}\right);$$

therefore

$$(1+mn)\left(\frac{du}{dx}\right)+(1-n)\left(\frac{du}{dy}\right)+(1+m)\left(\frac{du}{dz}\right) = -\{l+m+n+lmn\}\left(\frac{du}{dx}\right) = 0.$$

10. The radius of curvature at any point of

$$9a(a-x)^2 = (a+y)^2(a-2y)$$

varies as the square of the length of the normal intercepted between the point and the axis of abscissae.

$$\frac{dy}{dx} = \frac{3a(a-x)}{y(a+y)} = \frac{a}{y}\cdot\sqrt{\frac{a-2y}{a}};$$

$$\therefore \quad 1+\frac{dy^2}{dx^2} = \frac{(a-y)^2}{y^3}; \qquad -\frac{d^2y}{dx^2} = \frac{a(a-y)}{y^3};$$

$$\therefore \quad \frac{\sqrt{1+\frac{dy^2}{dx^2}}}{-y^2\frac{d^2y}{dx^2}} = \frac{1}{a}.$$

11. The involute of a circle rolls along a given straight line; prove that the centre of the circle will trace out a parabola.

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Take the given line as axis of x, and the starting-point as origin; then the coordinates of the centre of the circle are $x = s - a$; $y = p$;

$$\text{but } s = \text{length of arc of involute} = \frac{p^2}{2a};$$

$$\therefore \quad y^2 = 2a(x+a).$$

12. Find the envelope of the radical axis of two circles of given radius, the centre of one being fixed, and the centre of the other being always upon a fixed right line.

Take the given line as axis of x, and the perpendicular to it through the given point as axis of y;

then the equations of the circles are

$$(x-h)^2+y^2 = a^2; \qquad x^2+(y-k)^2 = b^2;$$

$\therefore$ their radical axis is

$$2hx-2ky+a^2-b^2-h^2+k^2 = 0;$$

differentiating with respect to h,

$$2x-h = 0;$$

$\therefore$ equation of envelope is

$$x^2-2ky+a^2-b^2+k^2 = 0.$$

VI.

Problems.

1. If each of two pairs of the equations

$$ax^2+bx+c = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$bx^2+cx+a = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$cx^2+ax+b = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

have a common root, shew that the third pair has also a common root; and if they all have a common root,

$$a^3+b^3+c^3-3abc = 0.$$

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If (1) and (2) have a common root,

$$\frac{x^3}{ab-c^3} = \frac{x}{bc-a^3} = \frac{1}{ca-b^3};$$

$$\therefore \quad (ab-c^3)(ca-b^3)^3 = (bc-a^3)^3.$$

So if (1) and (3) have a common root,

$$(ca-b^3)(bc-a^3)^3 = (ab-c^3)^3,$$

whence $(bc-a^3)(ab-c^3)^3 = (ca-b^3)^3,$

the condition that (2) and (3) should have a common root.

Also if the three have a common root

$$\begin{vmatrix} a, & b, & c \\ b, & c, & a \\ c, & a, & b \end{vmatrix} = 0;$$

or $$a^3+b^3+c^3-3abc = 0.$$

2. Two chords of a circle intersect at right angles. If the length of one of them be to the radius as $\sqrt{2}:1$, then the difference of the segments of the other is equal to the former.

Take EF parallel to CD through the centre,

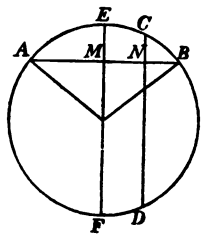
then $$DN-NC = FM-ME,$$

and since

$$AB = OA\sqrt{2},$$

$$\therefore \quad \angle AOB = \frac{\pi}{2};$$

$$\therefore \quad FM-ME = 2OM$$
$$= 2AM = AB.$$



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3. If $A+B+C=(2n+1)\pi$, shew that

$$u=\begin{vmatrix}\sin A, & \sin B, & \sin C\\ \cos A, & \cos B, & \cos C\\ \sin 3A, & \sin 3B, & \sin 3C\end{vmatrix}=0.$$

$$u=\begin{vmatrix}\sin A, & \sin B, & \sin C\\ \cos A, & \cos B, & \cos C\\ \sin A+\sin 3A, & \sin B+\sin 3B, & \sin C+\sin 3C\end{vmatrix}$$

$$=\begin{vmatrix}\sin A, & \sin B, & \sin C\\ \cos A, & \cos B, & \cos C\\ 4\sin A\cos^2 A, & 4\sin B\cos^2 B, & 4\sin C\cos^2 C\end{vmatrix}$$

$$=4\sin A\cos^2 A\sin(B-C)+4\sin B\cos^2 B\sin(C-A)$$
$$+4\sin C\cos^2 C\sin(A-B)$$

$$=(1+\cos 2A)(\cos 2C-\cos 2B)+(1+\cos 2B)(\cos 2A-\cos 2C)$$
$$+(1+\cos 2C)(\cos 2B-\cos 2A)=0.$$

4. In a spherical triangle, if

$$\cos a\cos\frac{c}{2}=\cos\left(b+\frac{c}{2}\right),\text{ then } A=2B.$$

$$\frac{\sin(A-B)}{\sin B}=\frac{1}{\sin b}\{\sin A\cos B-\sin b\cos A\}$$

$$=\frac{1}{\sin b}\left\{\frac{\cos b-\cos c-\cos a}{\sin c}-\frac{\cos a-\cos b\ \cos c}{\sin c}\right\}$$

$$=\frac{\cos b-\cos a}{\sin b}\cdot\frac{\sin c}{1-\cos c}=1 \quad \text{by question;}$$

$$\therefore \quad \sin(A-B)=\sin B,$$

$$\text{or} \quad A=2B.$$

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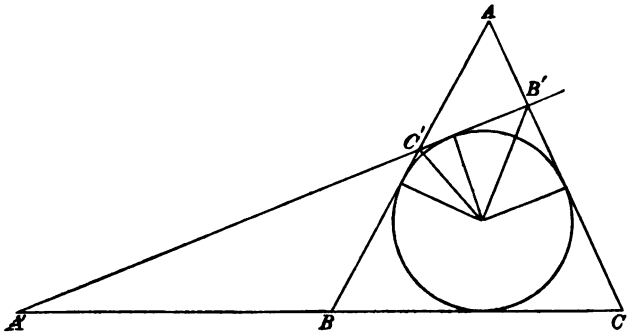
5. If from any point without a triangle ABC a tangent is drawn to the inscribed circle of which the radius is r, and α, β, γ are the portions of it intercepted opposite to the angles A, B, C respectively, then

$$\alpha^2 \tan\frac{A}{2} + \beta^2 \tan\frac{B}{2} + \gamma^2 \tan\frac{C}{2} = \frac{\alpha\beta\gamma}{r}.$$



Call $\angle C'A'B = \theta$;

then $\angle BC'B' = \pi - B + \theta$;

$\angle C'B'C = \pi - C - \theta$;

therefore

$$\alpha = B'C' = r\left\{\cot\frac{\pi - C - \theta}{2} + \cot\frac{\pi - B + \theta}{2}\right\} = r\frac{\cos\frac{A}{2}}{\cos\frac{B-\theta}{2}\cos\frac{C+\theta}{2}},$$

$$\beta = C'A' = r\left\{\cot\frac{\theta}{2} - \cot\frac{\pi - B + \theta}{2}\right\} = r\frac{\cos\frac{B}{2}}{\sin\frac{\theta}{2}\cos\frac{B-\theta}{2}},$$

$$\gamma = A'B' = r\left\{\cot\frac{\theta}{2} + \cot\frac{\pi - C - \theta}{2}\right\} = r\frac{\cos\frac{C}{2}}{\cos\frac{C+\theta}{2}\sin\frac{\theta}{2}};$$

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$$\therefore \quad \frac{a^2 \tan\frac{A}{2} + \beta^2 \tan\frac{B}{2} + \gamma^2 \tan\frac{C}{2}}{r^2}$$

$$= \tan\frac{A}{2} \cdot \left\{ \frac{\cos\frac{A}{2}}{\cos\frac{B-\theta}{2}\cos\frac{C+\theta}{2}} \right\}^2 + \tan\frac{B}{2} \left\{ \frac{\cos\frac{B}{2}}{\sin\frac{\theta}{2}\cos\frac{B-\theta}{2}} \right\}^2$$

$$+ \tan\frac{C}{2} \left\{ \frac{\cos\frac{C}{2}}{\cos\frac{C+\theta}{2}\sin\frac{\theta}{2}} \right\}^2$$

$$= \frac{\sin A(1-\cos\theta) + \sin B(1+\cos\overline{C+\theta}) + \sin C(1+\cos\overline{B-\theta})}{4\left\{\sin\frac{\theta}{2}\cos\frac{B-\theta}{2}\cos\frac{C+\theta}{2}\right\}^2}$$

$$= \frac{\sin A + \sin B + \sin C}{4\left\{\sin\frac{\theta}{2}\cos\frac{B-\theta}{2}\cos\frac{C+\theta}{2}\right\}^2}$$

$$= \frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\left\{\sin\frac{\theta}{2}\cos\frac{B-\theta}{2}\cos\frac{C+\theta}{2}\right\}^2} = \frac{a\beta\gamma}{r^2}.$$

6. *PQ*, *PR* are tangents to a parabola, such that $PQ.PR$ varies as SP^2, S being the focus; then the locus of P is a parabola having the same parameter as the given parabola.

Equation of parabola $y^2 = 4ax$.....(1),

. . . of straight line through P

$$\frac{x-x'}{l} = \frac{y-y'}{m} = r \text{.....(2)};$$

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$\therefore$ for lengths of tangents

$$m^2r^2+2(my'-2al)r+y'^2-4ax'=0;$$

$$\therefore \quad PQ.PR=\frac{y'^2-4ax'}{m_1m_2},$$

where m_1m_2 are the two values of m derived from the condition that (2) should be a tangent, viz.

$$(y'^2-4ax')m^2=(my'-2al)^2;$$

$$\frac{m^2}{l^2}-\frac{m}{l}\cdot\frac{y'}{x'}+\frac{y'^2}{4x'^2}=\frac{y'^2-4ax'}{4x'^2};$$

$$\therefore \quad \frac{m_1}{l_1}=\frac{y'+\sqrt{y'^2-4ax'}}{2x'}; \quad \frac{m_2}{l_2}=\frac{y'-\sqrt{y'^2-4ax'}}{2x'};$$

$$\therefore \quad \frac{l_1}{2x'}=\frac{m_1}{y'+\sqrt{y'^2-4ax'}}=\frac{1}{\{4x'^2+2y'^2-4ax'+2y'\sqrt{y'^2-4ax'}\}^{\frac{1}{2}}},$$

$$\frac{l_2}{2x'}=\frac{m_2}{y'-\sqrt{y'^2-4ax'}}=\frac{1}{\{4x'^2+2y'^2-4ax'-2y'\sqrt{y'^2-4ax'}\}^{\frac{1}{2}}};$$

$$\therefore \quad m_1m_2=\frac{ax'}{\left\{\{2x'^2+y'^2-2ax'\}^2-y'^2\{y'^2-4ax'\}\right\}^{\frac{1}{2}}}$$

$$=\frac{a}{2\{(x'-a)^2+y'^2\}^{\frac{1}{2}}}=\frac{a}{2SP};$$

$$\therefore \quad PQ.PR=k.SP=\frac{2SP}{a}\{y'^2-4ax'\};$$

$$\therefore \quad y'^2-4ax'-\frac{ka}{2}=0.$$

7. Through the extremity of the major axis of an ellipse a chord AP is drawn; the line joining the focus of the ellipse remote from A to the pole of AP cuts AP in Q; and a line through Q parallel to the minor axis meets the curve in R; shew that $(\cos ASQ)^2=\cos ASR$.

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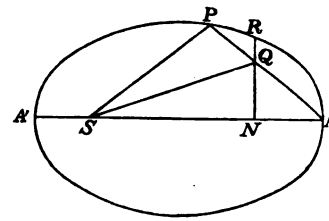
The angle subtended at the focus by any chord is bisected by the line joining the focus to the pole.



$\therefore$ if $\angle A'SP = \alpha$;

$$\angle ASQ = \frac{\pi - \alpha}{2},$$

and $\cos^2 ASQ = \sin^2 \frac{\alpha}{2}$.

Equation of AP is $\frac{l}{r} = e\cos\theta + \operatorname{cosec}\frac{\alpha}{2}\sin\left(\theta - \frac{\alpha}{2}\right)$;

$\therefore$ putting $\theta = \frac{\pi + \alpha}{2}$,

we have $SQ = \dfrac{l}{\operatorname{cosec}\frac{\alpha}{2} - e\sin\frac{\alpha}{2}}$, and $SN = \dfrac{l\sin\frac{\alpha}{2}}{\operatorname{cosec}\frac{\alpha}{2} - e\sin\frac{\alpha}{2}}$;

$\therefore$ equation of QN is $\frac{l}{r} = \cos\theta\left(e - \operatorname{cosec}^2\frac{\alpha}{2}\right)$;

for point of intersection with ellipse $\frac{l}{r} = 1 + e\cos\theta$,

we have $1 + \cos\theta\cos ec^2\frac{\alpha}{2} = 0$,

or $\cos ASR = \sin^2\frac{\alpha}{2}$.

8. A parabola and its circle of curvature at any point have in general only one chord of intersection. Prove this, and shew that the points where such chords are met by perpendiculars from the vertex lie in a cissoid.

If a circle intersect a conic, its chords of intersection make equal angles with the axis.

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If the circle is the circle of curvature, three points of intersection coincide; the tangent at the point is one chord of intersection, and there is but one other; and this and the tangent make equal angles with the axis: the equation therefore of this chord in the case of the parabola is

$$x.2a+yy'-\frac{3}{2}y'^2=0,$$

and equation of perpendicular from the vertex is

$$x.y'-y.2a=0:$$

for point of intersection

$$\frac{x}{2a}=\frac{y}{y'}=\frac{\frac{3}{2}y'^2}{y'^2+4a^2}=\left\{\frac{x^2+y^2}{y'^2+4a^2}\right\}^{\frac{1}{2}};$$

$$\therefore\quad \frac{3}{2}yy'=x^2+y^2,$$

$$\text{or}\quad \frac{3ay^2}{x}=x^2+y^2;$$

$$\therefore\quad y^2=\frac{x^3}{3a-x}.$$

9. If ω is the angle between the axes of coordinates, then the equation of the directrix of the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$$

is $x(a+b\cos\omega)+y(a\cos\omega+b)=ab\cos\omega.$

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

equation of conic inscribed in a triangle whose sides are the two axes, and the line at infinity:

$$\left.\begin{aligned} lx+my&=1\\ l'x+m'y&=1\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

any two straight lines.

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Conditions that these should touch (1) are

$$\left.\begin{aligned}\frac{1}{al}+\frac{1}{bm}-1=0\\ \frac{1}{al'}+\frac{1}{bm'}-1=0\end{aligned}\right\} \text{ or } \left\{\begin{aligned}al+bm-albm=0\\ al'+bm'-abl'm'=0\end{aligned}\right\} \ldots(3);$$

condition that they should be at right angles is

$$ll'+mm'-(lm'+l'm)\cos\omega=0 \ldots\ldots\ldots\ldots\ldots(4):$$

then the locus of the intersection of (2) is the directrix.

From (2), $\dfrac{\varpi}{m'-m}=\dfrac{y}{l-l'}=\dfrac{1}{lm'-l'm}$,

from (3), $\dfrac{a}{mm'(l-l')}=\dfrac{b}{ll'(m'-m)}=\dfrac{ab}{lm'-l'm}$;

$$\therefore \quad \frac{a}{mm'y}=\frac{b}{ll'x}=ab \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Again, from (2),

$$ll'x^2+mm'y^2+(lm'+l'm)xy=1;$$

$\therefore$ from (4),

$$ll'x\left(x+\frac{y}{\cos\omega}\right)+mm'y\left(\frac{x}{\cos\omega}+y\right)=1;$$

$\therefore$ from (5),

$$b\left(x+\frac{y}{\cos\omega}\right)+a\left(\frac{x}{\cos\omega}+y\right)=ab,$$

or $$x(a+b\cos\omega)+y(a\cos\omega+b)=ab\cos\omega.$$

10. A given ellipse slides between two given straight lines at right angles to each other; shew that either focus will trace out the curve

$$4(a^2-b^2)=\left(\frac{b^2-x^2}{x}\right)^2+\left(\frac{b^2-y^2}{y}\right)^2,$$

$2a$, $2b$ being the axes of the ellipse, and the given straight lines the axes of coordinates.

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Let (x_1, y_1), (x_2, y_2) be the foci;

then coordinates of centre are $\frac{x_1+x_2}{2}$, $\frac{y_1+y_2}{2}$.

Now square of distance of centre of an ellipse from point of intersection of rectangular tangents = sum of squares of semi-axes;

$$\therefore \quad (x_1+x_2)^2+(y_1+y_2)^2 = 4(a^2+b^2),$$

$$\text{but} \quad x_1x_2 = y_1y_2 = b^2;$$

$$\therefore \quad \left\{x+\frac{b^2}{x}\right\}^2+\left\{y+\frac{b^2}{y}\right\}^2 = 4(a^2+b^2),$$

$$\text{whence} \quad \left\{\frac{b^2-x^2}{x}\right\}^2+\left\{\frac{b^2-y^2}{y}\right\}^2 = 4(a^2-b^2).$$

11. The focal distances of three points of a parabola are in arithmetic progression; shew that the normals at these points intersect in a semi-cubical parabola.

Equation of normal at (x_1y_1) is

$$x.y_1+y.2a-y_1(x_1+2a) = 0;$$

equation of normal at (x_2y_2) is

$$x.y_2+y.2a-y_2(x_2+2a) = 0;$$

for intersection

$$\frac{x}{y_1(x_1+2a)-y_2(x_2+2a)} = \frac{2ay}{y_1y_2(x_2+2a)-y_1y_2(x_1+2a)} = \frac{1}{y_1-y_2};$$

$$\therefore \quad 4ax = 8a^2+y_1^2+y_1y_2+y_1^2; \quad -8a^2y = y_1y_2(y_1+y_2).$$

Now by well-known condition for three normals to a parabola meeting in a point

$$y_1+y_2+y_3 = 0,$$

and by prob. $\quad y_1^2+y_3^2 = 2y_2^2;$

whence $\quad 2y_1(y_1+y_2) = y_2^2;$

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$$\therefore \quad 4ax = 8a^2 + \frac{3}{2}y_2^2; \quad -8a^2y = \frac{1}{2}y_2^3;$$

$$\therefore \quad y^3 = \frac{2}{27a}(x-2a)^3.$$

12. Find $f(x)$ when $f(x)f(y)-f(xy) = x+y$, x and y being independent.

put $y = o$, $\therefore f(x)f(o)-f(o) = x$, or $f(o) = \dfrac{x}{f(x)-1}$,

i. e. $\dfrac{x}{f(x)-1}$ is independent of x;

this suggests $f(x) = 1+kx$;

and it is found that k must $= 1$.

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I.

Algebra.

1. **Prove that**

$$(x+y+z)^{2n+1}-x^{2n+1}-y^{2n+1}-z^{2n+1}$$

is divisible by $(y+z)\,(z+x)\,(x+y)$.

$$(x+y+z)^{2n+1}-x^{2n+1}-y^{2n+1}-z^{2n+1}$$
$$= x^{2n}(y+z)+(2n+1)x^{2n-1}(y+z)^2+\cdots-(y^{2n+1}+z^{2n+1}),$$

and this is clearly divisible by $(y+z)$;

similarly the given expression is divisible by $(z+x)$ and $(x+y)$, and therefore by the product of the three.

2. **Extract the square root of**

(1) $(\sqrt{3}+\sqrt{2})^2-4\sqrt[4]{6}\,(\sqrt{3}-\sqrt{2})$;

(2) $2a+2\sqrt{a^2-b^2-c^2+2bc}$.

(1) $\{\sqrt{3}+\sqrt{2}\}^2-4\sqrt[4]{6}\,\{\sqrt{3}-\sqrt{2}\}$
$= \{\sqrt{3}-\sqrt{2}\}^2-2.2\sqrt[4]{6}\,\{\sqrt{3}-\sqrt{2}\}+\{2\sqrt[4]{6}\}^2$
$= \{\sqrt{3}-\sqrt{2}-2\sqrt[4]{6}\}^2$;

(2) $2a+2\sqrt{a^2-b^2-c^2+2bc} = \{\sqrt{a+b-c}+\sqrt{a-b+c}\}^2$.

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3. Solve the equations:

(1) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 2$ .....(α),

$\sqrt{\frac{x}{z}} + \sqrt{\frac{z}{x}} = 3$.....($\beta$),

$\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}} = 4\sqrt{x}$.....($\gamma$);

(2) $\frac{16x^4 - 8x^2 + 1}{8x^3 + 4x^2 + 2x + 1} = x - \frac{1}{2}$;

(4) $\frac{1 - \sqrt{1-x^2}}{1 + \sqrt{1-x^2}} = a\,\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$;

(5) $2x^3 + 27x^2 = 54$.

(1) from (α), $x - 2\sqrt{xy} + y = 0$;

$\therefore\ x = y = \frac{9}{16}$ from (β) and (γ),

and $z = \frac{63 + 27\sqrt{5}}{32}$;

(2) $2(4x^2-1)^2 = (4x^2+1)(4x^2-1)$;

$\therefore\ 4x^2 - 1 = 0$, or $4x^2 - 3 = 0$;

(4) $\frac{\{\sqrt{1+x} - \sqrt{1-x}\}^2}{\{\sqrt{1+x} + \sqrt{1-x}\}^2} = a \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$;

$\therefore\ \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \sqrt[3]{a}$, or $a\sqrt[3]{a}$, or $a^2\sqrt[3]{a}$,

where 1, a, a^2 are the 3 cube roots of 1;

taking first value, $\frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{1 + \sqrt[3]{a}}{1 - \sqrt[3]{a}}$;

whence $x = \frac{2\sqrt[3]{a}}{1 + \sqrt[3]{a^2}}$.

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(5) $(2x+3)(x^2+12x-18)=0$,

whence the roots are $-\frac{3}{2}$, $-6\pm3\sqrt{6}$.

5. Expand $\frac{x}{\sqrt{(1+x)}}$ in a series proceeding by ascending powers of $\frac{x}{1+x}$.

$$\frac{x}{\sqrt{1+x}}=\left\{x-\frac{x}{1+x}\right\}^{\frac{1}{2}}.$$

6. Sum the following series to n terms:

(1) $1+x+x^2+2x^3+2x^4+5x^5+4x^6+\ldots$,

(2) $\frac{3}{5.8.11}+\frac{6}{8.11.14}+\frac{9}{11.14.17}+\ldots$.

(1) A recurring series, the scale of relation being

$$1-3x^2+x^3;$$

$$S_\infty=\frac{1+x-2x^2}{1-3x^2+x^3};$$

the nth and two preceding terms have now to be found by expansion (previously breaking up into partial fractions where praticable);
and thus S_n can be found.

(2) $(n+1)$th term $=\frac{3n+3}{(3n+5)(3n+8)(3n+11)}$

$$=\frac{1}{(3n+5)(3n+8)}-\frac{8}{(3n+5)(3n+8)(3n+11)}$$

$$=(3n+5)^{(-2)}-8(3n+5)^{(-3)};$$

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$$\therefore \quad S_n = \Sigma.(3n+5)^{(-2)} - 8\Sigma(3n+5)^{(-3)}$$

$$= -\frac{(3n+5)^{(-1)}}{3} + \frac{4(3n+5)^{(-2)}}{3} + C$$

$$= -\frac{1}{3(3n+5)} + \frac{4}{3(3n+5)(3n+8)} + C$$

$$= -\frac{3n+4}{3(3n+5)(3n+8)} + C;$$

$$\text{put } n = 0; \qquad \therefore \quad C = \frac{1}{30}.$$

7. Find the number of ways in which m things can be distributed into n compartments, n being less than m, and it being understood that no compartment is to be left altogether empty.

Place one in each compartment, then $(m-n)$ things are left to be disposed of;
and $(m-n)$ things can be distributed into n compartments in n^{m-n} ways.

8. Shew that any number of pounds can be weighed with the series of weights 1 lb., 3 lbs., 9 lbs., 27 lbs., &c.

Any number N, expressed in the scale whose radix is 3, is of the form $p_0 + p_1 3 + p_2 3^2 + \dots$, where each of the quantities $p_0, p_1, p_2 \dots$ is 0, 1, or 2.

Now, for example, let $N = 1 + 2.3 + 2.3^2 + 3^3$

$$= 1 + (3-1)3 + (3-1)3^2 + 3^3;$$

$$\therefore \quad N + 3 = 1 + 2.3^3 = 1 + (3-1)3^3;$$

$$\therefore \quad N + 3 + 3^3 = 1 + 3^4;$$

thus only one weight of each kind is used; but they are put in either scale. Compare question 12.

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11. Prove the following:

(1) The 18th power of any number not divisible by 3 is of the form $27m+1$;

(2) If n be any integer, $3^{2n}-8n-1$ is divisible by 64.

(1) $(3n \pm 1)^{18} = 27m+1$;

(2) $(8+1)^n-8n-1 = 64m$.

12. Find the series equivalent to the product

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots \text{ ad inf.}$$

$$(1+x)(1+x^2)(1+x^4)\ldots = \frac{1}{1-x}(1-x)(1+x)(1+x^2)(1+x^4)$$

$$= \frac{1-x^\infty}{1-x} = 1+x+x^2+x^3+x^4+\ldots.$$

NOTE.—This equivalence may be used to shew that any number of pounds can be weighed with the series of weights 1 lb., 2 lbs., 4 lbs., 8 lbs., &c., only one of each kind being used, and the weights being put only in one scale; for, in multiplying out the product, the indices 1, 2, 4, &c. are added together, and the series shews that any number whatever is so obtained. Compare question 8.

13. If $c_0, c_1, c_2 \ldots\ldots c_n$ be the coefficients in the expansion of $(1+x)^n$, shew that

$$\frac{c_0}{1.2}+\frac{c_1}{2.3}+\frac{c_2}{3.4}+\ldots\ldots+\frac{c_n}{n+1\,n+2} = \frac{2^{n+2}-n-3}{(n+1)(n+2)}.$$

$$\frac{c_0}{1.2}+\&c. = \frac{1}{(n+1)(n+2)}\left\{\frac{(n+2)(n+1)}{1.2}\right.$$

$$\left. + \frac{(n+2)(n+1)n}{1.2.3} + \ldots\ldots + (n+2)+1\right\}$$

$$= \frac{1}{(n+1)(n+2)}\left\{1+(n+2)+\frac{(n+2)(n+1)}{1.2}+\ldots\right.$$

$$\left.\ldots+(n+2)+1-(n+3)\right\} = \frac{2^{n+2}-n-3}{(n+1)(n+2)}.$$

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Or thus—

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_n x^n;$$

therefore, integrating twice, and adding the necessary constants,

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{c_0 x^2}{1.2} + \frac{c_1 x^3}{2.3} + \ldots + \frac{c^n x^{n+2}}{(n+1)(n+2)} + \frac{x}{n+1} + \frac{1}{(n+1)(n+2)};$$

put $x = 1$;

therefore

$$\frac{2^{n+2}}{(n+1)(n+2)} = \frac{c_0}{1.2} + \frac{c_1}{2.3} + \ldots + \frac{c_n}{(n+1)(n+2)} + \frac{n+3}{(n+1)(n+2)}.$$

15. A man has left his umbrella in one of three shops, which he visited in succession. He is in the habit of leaving it, on an average, once in every four times that he goes to a shop. Find the chance of his having left it in the first, second, and third shops, respectively.

Chance that he left it in the first shop is proportional to $\frac{1}{4}$;

chance that he did not leave it in the first, but did in the second, is proportional to . . . . . $\frac{3}{4}.\frac{1}{4} = \frac{3}{16}$;

chance that he did not leave it in the first or second, but did in the third, is proportional to . . $\frac{3}{4}.\frac{3}{4}.\frac{1}{4} = \frac{9}{64}$.

Now the umbrella was left;

∴ the actual chances are

$$\frac{\frac{1}{4}}{\frac{1}{4}+\frac{3}{16}+\frac{9}{64}}, \quad \frac{\frac{3}{16}}{\frac{1}{4}+\frac{3}{16}+\frac{9}{64}}, \quad \frac{\frac{9}{64}}{\frac{1}{4}+\frac{3}{16}+\frac{9}{64}} \text{ respectively;}$$

or $\frac{16}{37}, \frac{12}{37}, \frac{9}{37}$.

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II.

Trigonometry and Theory of Equations.

3. If a, b, c be the sides of a triangle opposite to the angles A, B, C, and r, r_a, r_b, r_c the radii of the inscribed and escribed circles, then

$$(1)\quad \frac{\cos\frac{B-C}{2}}{\cos\left(60°-\frac{B}{2}\right)\cos\left(60°-\frac{C}{2}\right)} + \frac{\cos\frac{C-A}{2}}{\cos\left(60°-\frac{C}{2}\right)\cos\left(60°-\frac{A}{2}\right)} + \frac{\cos\frac{A-B}{2}}{\cos\left(60°-\frac{A}{2}\right)\cos\left(60°-\frac{B}{2}\right)} = 4;$$

$$(2)\quad 2r = \frac{a\tan\frac{B}{2}\tan\frac{C}{2} + b\tan\frac{C}{2}\tan\frac{A}{2} + c\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}};$$

$$(3)\quad 1 = \frac{ab}{(r_b+r_c)(r_c+r_a)} + \frac{bc}{(r_c+r_a)(r_a+r_b)} + \frac{ca}{(r_a+r_b)(r_b+r_c)}.$$

(1) $2\cos\frac{B-C}{2}\cos\left(60°-\frac{A}{2}\right) = \sin(120°-B)+\sin(120°-C);$

$\therefore$ given expression

$$= \frac{\sin(120°-A)+\sin(120°-B)+\sin(120°-C)}{\cos\left(60°-\frac{A}{2}\right)\cos\left(60°-\frac{B}{2}\right)\cos\left(60°-\frac{C}{2}\right)} = 4.$$

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$$(2)\quad r\tan\frac{A}{2}=\frac{(s-b)(s-c)}{s};$$

$$\therefore\quad 2r\left\{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right\}$$

$$=2\cdot\frac{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)}{s}$$

$$=\frac{6s^2-4(a+b+c)s+2(bc+ca+ab)}{s}=\frac{2s^2-(a^2+b^2+c^2)}{s}$$

$$=\frac{a(s-a)+b(s-b)+c(s-c)}{s}$$

$$=a\tan\frac{B}{2}\tan\frac{C}{2}+b\tan\frac{C}{2}\tan\frac{A}{2}+c\tan\frac{A}{2}\tan\frac{B}{2}.$$

$$(3)\quad r_b+r_c=a\cot\frac{A}{2};$$

$$\therefore\ \text{expression}=\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

6. . . . . Shew that

$$1^2+2^2+\frac{3^2}{1.2}+\frac{4^2}{.2.3}+\ldots\ldots=5e.$$

$$n\text{th term}=\frac{n^2}{\lfloor n-1}=\frac{(n-1)(n-2)+3(n-1)+1}{\lfloor n-1}$$

$$=\frac{1}{\lfloor n-3}+\frac{3}{\lfloor n-2}+\frac{1}{\lfloor n-1};$$

$$\therefore\quad S=e+3e+e=5e.$$

7. Prove that

$$1-3\cos\theta+5\cos 2\theta-7\cos 3\theta+\ldots\text{ ad inf.}$$
$$+2\,(3\sin\theta-5\sin 2\theta+7\sin 3\theta-\ldots\text{ ad inf.})^2=0.$$

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$$C = 3\cos\theta - 5\cos 2\theta + 7\cos 3\theta - \ldots,$$
$$S = 3\sin\theta - 5\sin 2\theta + 7\sin 3\theta - \ldots;$$

$$\therefore\ C + S\sqrt{-1} = 3\{\cos\theta + \sqrt{-1}\sin\theta\} - 5\{\cos\theta + \sqrt{-1}\sin\theta\}^2 + \ldots$$
$$= \frac{3\{\cos\theta + \sqrt{-1}\sin\theta\} + \{\cos\theta + \sqrt{-1}\sin\theta\}^2}{\{1 + \cos\theta + \sqrt{-1}\sin\theta\}^2};$$

$$= \frac{\{3\cos\theta + \cos 2\theta + \sqrt{-1}(3\sin\theta + \sin 2\theta)\}\{1 + \cos\theta - \sqrt{-1}\sin\theta\}^2}{4(1+\cos\theta)^2}$$

$$C = \frac{((1+\cos\theta)^2 - \sin^2\theta)(3\cos\theta + \cos 2\theta) + 2\sin\theta(1+\cos\theta)(3\sin\theta + \sin 2\theta)}{4(1+\cos\theta)^2}$$
$$= \frac{\cos^2\theta + 4\cos\theta + 3}{2(1+\cos\theta)^2};$$

$$S = \frac{((1+\cos\theta)^2 - \sin^2\theta)(3\sin\theta + \sin 2\theta) - 2\sin\theta(1+\cos\theta)(3\cos\theta + \cos 2\theta)}{4(1+\cos\theta^2)}$$
$$= \frac{\sin\theta}{2(1+\cos\theta)};$$

$$\therefore\quad 1 - C + 2S^2 = -\frac{1-\cos^2\theta}{2(1+\cos\theta)^2} + \frac{\sin^2\theta}{2(1+\cos\theta)^2} = 0.$$

8. If a, b, c be the sides of a spherical triangle opposite to the angles A, B, C, then

$$\cot A = \frac{\cos b\cos c - \cot B\cot C}{\cos b\cot B + \cos c\cot C};$$

also deduce the corresponding property of a plane triangle.

By known formulae, $\cos c = \dfrac{\cos C + \cos A\cos B}{\sin A\sin B}$,

$$\sin C\cos b = \cos A\sin B\cos c + \sin A\cos B;$$

$$\therefore\quad \cos b(\cot A\cot B - \cos c) = \frac{-\cos b\cos C}{\sin A\sin B};$$

$$\cot C(\cos c\cot A + \cot B)$$
$$= \frac{\cos C}{\sin A\sin B\sin C}(\cos A\sin B\cos c + \sin A\cos B) = \frac{\cos b\cos C}{\sin A\sin B};$$

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$$\therefore \quad \cos b\,(\cot A \cot B - \cos c) = -\cot C\,(\cos c \cot A + \cot B);$$

$$\therefore \quad \cot A = \frac{\cos b \cos c - \cot B \cot C}{\cos b \cot B + \cos c \cot C}.$$

To prove the corresponding formula for a plane triangle, put $\frac{a}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$, for a, b, c.

$$\text{then} \quad \cot A = \frac{\left\{1 - \frac{\beta^2}{2r^2} + \dots\right\}\left\{1 - \frac{\gamma^2}{2r^2} + \dots\right\} - \cot B \cot C}{\left\{1 - \frac{\beta^2}{2r^2} + \dots\right\}\cot B + \left\{1 - \frac{\gamma^2}{2r^2} + \dots\right\}\cot C};$$

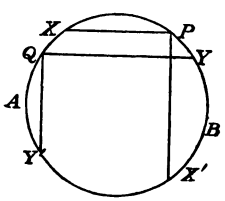
make r infinite,

$$\therefore \quad \cot A = \frac{1 - \cot B \cot C}{\cot B + \cot C}.$$

III.

Pure Geometry.

10. AX and BY are equal arcs of a circle measured in opposite directions from two fixed points A and B on the circumference. If P and Q are two other fixed points on the same circumference, prove that the locus of the intersection of PX, QY is an equilateral hyperbola; and determine the directions of its asymptotes.



P, Q will evidently be centres of two homographic pencils, which are not in perspective, since PQ will not in general correspond to QP;

the locus therefore of the intersection of PX, QY is a conic;

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also it is evident that there are two positions of the points X, Y in which arc XY = arc PQ.

PX, QY are then parallel;

thus the conic has points at infinity in two directions, and is therefore a hyperbola;

PX, PX' in the figure are the directions of the asymptotes.

Also arc $XAX' = XA + AY' + Y'X'$

$$= YB + BX' + XY = \text{arc } XBX';$$

$\therefore$ PX, PX' are at right angles.

14. Tangents from the vertices of a triangle ABC are drawn to a conic section, and meet the opposite sides in the points $a_1\ a_2$, $b_1\ b_2$, $c_1\ c_2$ respectively. Prove that a conic section can be drawn through these six points.

By the reciprocal of Carnot's theorem,

$$\frac{\sin CBb_1 . \sin CBb_2}{\sin ABb_1 . \sin ABb_2} \cdot \frac{\sin ACc_1 . \sin ACc_2}{\sin BCc_1 . \sin BCc_2} \cdot \frac{\sin BAa_1 . \sin BAa_2}{\sin CAa_1 . \sin CAa_2} = 1;$$

this may be replaced by

$$\frac{Cb_1 . Cb_2}{Ab_1 . Ab_2} \cdot \frac{Ac_1 . Ac_2}{Bc_1 . Bc_2} \cdot \frac{Ba_1 . Ba_2}{Ca_1 . Ca_2} = 1;$$

$\therefore$ by Carnot's theorem the six points a_1, a_2, b_1, b_2, c_1, c_2 lie on a conic.

IV.

Analytical Geometry.

2. Under what conditions will $\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = 2kxy$ represent a circle? If ω be the inclination of the axes, shew that the radius $= a \tan \frac{\omega}{2}$.

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$x = 0$, $y = 0$ are tangents, and $\frac{x}{a} + \frac{y}{b} - 1 = 0$ the chord of contact;

for a circle $a = b$ ..... (1);

thus equations for finding the centre become

$$x + (1 - 2ka^2)y - a = 0,$$

$$(1 - 2ka^2)x + y - a = 0;$$

$$\therefore \quad \frac{x}{ka^2} = \frac{y}{ka^2} = \frac{a^2}{1 - ka^2},$$

but clearly $r = a \tan \frac{\omega}{2} = y \sin \omega$;

$$\therefore \quad a = 2y \cos^2 \frac{\omega}{2} = \frac{2ka^2 \cos^2 \frac{\omega}{2}}{1 - ka^2};$$

$$\therefore \quad k = \frac{1}{a^2 \cos \omega} \text{ ..... (2).}$$

4. The directrix of the parabola

$$(\sqrt{a}.x + \sqrt{c}.y)^2 + 2dx + 2ey + \frac{d^2 + e^2}{a + c} = 0$$

is given by $\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{c}} = 0$; and the focus by

$$\frac{x}{d} = \frac{y}{e} = -\frac{1}{a + c}.$$

If $(x_1\, y_1)$ is the focus, and $Ax + By + C = 0$ the directrix, the equation of the parabola is

$$(x - x_1)^2 + (y - y_1)^2 = \frac{\{Ax + By + C\}^2}{A^2 + B^2};$$

and given equation may be written in the form

$$\left\{x + \frac{d}{a + c}\right\}^2 + \left\{y + \frac{e}{a + c}\right\}^2 = \frac{\left\{\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{c}}\right\}^2}{\frac{1}{a} + \frac{1}{c}};$$

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$$\therefore \text{ focus is given by } \quad x_1 = -\frac{d}{a+c}; \quad y_1 = -\frac{e}{a+c};$$

$$\text{and directrix is } \quad \frac{x}{\sqrt{a}} - \frac{y}{\sqrt{c}} = 0.$$

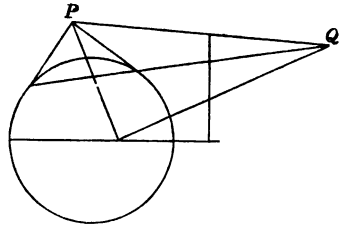
6. If from any point in the plane of an ellipse a perpendicular be let fall upon its polar, the two confocal conics which can be drawn through the foot of the perpendicular will touch at that point the one the polar, the other the perpendicular.

To prove this by reciprocation—

A system of circles having the same radical axis reciprocates into a system of confocal conics, one of the limiting points being taken as origin: also lines at right angles become points subtending a right angle at the origin:

thus the problem becomes this—

if P be any point in the plane of a circle belonging to a coaxal system, and Q a point on its polar such that PQ subtends a right angle at a limiting point, then PQ is a tangent to the two circles of the system that can be drawn through P or Q.



Now the polars of any point P with respect to a system of circles having the same radical axis all pass through the opposite extremity of the diameter of the orthogonal circle passing through P;

also all orthogonal circles pass through the limiting points;

$\therefore$ PQ is clearly a diameter of a circle orthogonal to the system;

and $\therefore$ touches either of the circles of the system drawn through P or Q.

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7. Normals are drawn to a parabola from a given point, and perpendiculars are let fall on them from the focus; shew that the sum of the squares of the perpendiculars $= 2(x-m)(x-2m)$, x being the abscissa of the point measured from the vertex, and $4m$ the latus rectum.

Square of perpendicular from focus on normal at any point $P = SP^2 - SY^2 = SP^2 - AS.SP = x'(x'+m)$;

$\therefore$ sum of squares $= \Sigma x'^2 + m\Sigma x'$.

Now equation of normal at any point $(x'\ y')$ is

$$y'.x + 2my - (2m+x')y' = 0;$$

$\therefore$ substituting for y' in the equation of the parabola

$$x'^2 + 2(2m-x)x' + (2m-x)^2.x' - my^2 = 0;$$

whence $\Sigma x' = -2(2m-x)$; $\quad \Sigma x'^2 = 2(2m-x)^2$;

$$\therefore \quad \Sigma x'^2 + m\Sigma x' = 2(x-m)(x-2m).$$

8. From a given point as centre circles are described cutting an ellipse. Shew that the locus of the middle points of the common chords of the circles and ellipse is an equilateral hyperbola, passing through the feet of the normals drawn from the given point to the ellipse.

Let $(x_1\ y_1)$ be the given point;

then the equation of the normal at any point $(x'\ y')$ on an ellipse is

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = a^2 - b^2;$$

but if $(x\ y)$ is known $(x_1\ y_1)$; and $(x'\ y')$ unknown, this becomes

$$\frac{a^2x_1}{x} - \frac{b^2y_1}{y} = a^2 - b^2,$$

or $\quad (a^2-b^2)xy + b^2y_1x - a^2x_1y = 0;$

and this represents an equilateral hyperbola passing through the feet of the normals.

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Again, the equation of a chord of the circle

$$(x-x_1)^2+(y-y_1)^2 = r^2,$$

in terms of its middle point (x_2, y_2) is

$$x(x_1-x_2)+y(y_1-y_2) = x_2(x_1-x_2)+y_2(y_1-y_2);$$

and the equation of the same chord for the ellipse is

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2} = \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2};$$

$$\therefore \quad \frac{x_1-x_2}{\frac{x_2}{a^2}} = \frac{y_1-y_2}{\frac{y_2}{b^2}};$$

$$\therefore \quad (a^2-b^2)x_2y_2+b^2y_1x_2-a^2x_1y_2 = 0;$$

shewing that (x_2, y_2) lies on the hyperbola determined above.

10. Perpendiculars from the angles of a triangle meet the opposite sides in D_1, D_2, D_3; if O_1, O_2, O_3 be the centres of the escribed circles, shew that D_1O_1, D_2O_2, D_3O_3 meet in a point which lies in the straight line joining the centres of the inscribed and circumscribed circles.

Equation of D_1O_1 passing through $(-1, 1, 1)$ and $\{0, \cos C, \cos B\}$ is

$$\alpha(\cos C-\cos B)-\beta\cos B+\gamma\cos C = 0;$$

so for D_2O_2, $\quad \alpha\cos A+\beta(\cos A-\cos C)-\gamma\cos C = 0;$

and for D_3O_3, $\quad -\alpha\cos A+\beta\cos B+\gamma(\cos B-\cos A) = 0:$

for line joining centres of inscribed and circumscribed circles

$$\{1, 1, 1\} \text{ and } \{\cos A, \cos B, \cos C\};$$

$$\alpha\{\cos B-\cos C\}+\beta\{\cos C-\cos A\}+\gamma\{\cos A-\cos B\} = 0:$$

any three of these clearly meet in a point.

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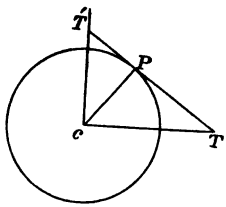
14. If the coordinates of a straight line be the reciprocals of its intercepts on two rectangular axes, find the tangential equation of a circle, of which the origin is the centre.



$$CP^2.TT'^2 = CT^2.CT'^2;$$

$$\therefore \quad \frac{1}{CT^2} + \frac{1}{CT'^2} = \frac{1}{CP^2},$$

$$\text{or} \qquad x^2+y^2 = \frac{1}{a^2}.$$

V.

Differential Calculus.

8. Find the equation of the tangent to the curve

$$(a+b)\,y = \frac{a^3}{x-a} - \frac{b^3}{x-b}$$

at its point of inflexion.

$$(a+b)\frac{dy}{dx} = -\frac{a^3}{(x-a)^2} + \frac{b^3}{(x-b)^2};$$

$$(a+b)\frac{d^2y}{dx^2} = \frac{2a^3}{(x-a)^3} - \frac{2b^3}{(x-b)^3} = 0$$

at a point of inflexion;

$\therefore$ the point of inflexion is

$$x = 0; \qquad y = -a+b:$$

$$\therefore \quad \frac{dy}{dx} \text{ at the point } = -\frac{a-b}{a+b};$$

$\therefore$ the equation of the tangent is

$$(a-b)x+(a+b)y+a^2-b^2 = 0.$$

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11. . . . If ρ_1, ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola, of which $4m$ is the latus rectum, then $(\rho_1 \rho_2)^{\frac{2}{3}} = (2m)^{\frac{2}{3}} (\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}})$.

$$\rho_1 = \frac{2(m+x_1)^{\frac{3}{2}}}{m^{\frac{1}{2}}}; \qquad \rho_2 = \frac{2(m+x_2)^{\frac{3}{2}}}{m^{\frac{1}{2}}};$$

$$\therefore \quad (2m)^{\frac{2}{3}} (\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}}) = 2^{\frac{4}{3}} m^{\frac{1}{3}} (2m + x_1 + x_2);$$

$$(\rho_1 \rho_2)^{\frac{2}{3}} = 2^{\frac{4}{3}} m^{-\frac{2}{3}} (m^2 + m(x_1 + x_2) + x_1 x_2)$$

$$= 2^{\frac{4}{3}} m^{\frac{1}{3}} (2m + x_1 + x_2),$$

since $x_1 x_2 = m^2$ for the extremities of a focal chord.

VI.

Problems.

2. Any polygonal number of order r, multiplied by $8\,\frac{(r-1)}{(r-3)^2}$, and increased by unity, makes a square number.

$$\left\{n + \frac{n(n-1)(r-1)}{2}\right\} \frac{8(r-1)}{(r-3)^2} + 1 = \frac{4n^2(r-1)^2}{(r-3)^2} - \frac{4n(r-1)}{r-3} + 1$$

$$= \left\{\frac{2n(r-1)}{r-3} - 1\right\}^2.$$

3. The sides of a triangle are such that

$$\frac{a}{1+m^2n^2} = \frac{b}{m^2+n^2} = \frac{c}{(1-m^2)(1+n^2)};$$

shew that the angles opposite to a and b are

$$2\tan^{-1}\frac{m}{n} \text{ and } 2\tan^{-1}mn,$$

and that the area is $\frac{mn}{m^2+n^2}\,bc$.

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$$\frac{\sin A}{1+m^2n^2} = \frac{\sin B}{m^2+n^2} = \frac{\sin(A+B)}{(1-m^2)(1+n^2)} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{(1+m^2)(1+n^2)}$$

$$= \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{(1-m^2)(1-n^2)} = \frac{2\sin\frac{A+B}{2}\cos\frac{A+B}{2}}{(1-m^2)(1+n^2)};$$

$$\therefore \quad \frac{\cos\frac{A-B}{2}}{1+m^2} = \frac{\cos\frac{A+B}{2}}{1-m^2} = \frac{\cos\frac{A}{2}\cos\frac{B}{2}}{1} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{m^2},$$

$$\frac{\sin\frac{A-B}{2}}{1-n^2} = \frac{\sin\frac{A+B}{2}}{1+n^2} = \frac{\sin\frac{A}{2}\cos\frac{B}{2}}{1} = \frac{\cos\frac{A}{2}\sin\frac{B}{2}}{n^2};$$

$$\therefore \quad \tan\frac{A}{2} = \frac{m}{n}; \qquad \tan\frac{B}{2} = mn;$$

whence also $\sin A = \dfrac{2mn}{m^2+n^2}$;

$$\therefore \quad \text{area} = \frac{mn}{m^2+n^2}bc.$$

4. The nth fraction convergent to

$$\frac{1}{2\cos\theta-}\ \frac{1}{2\cos\theta-}\ \frac{1}{2\cos\theta-}\ \ldots\ldots \text{ is } \frac{\sin n\theta}{\sin(n+1)\theta}.$$

$$1\text{st convergent} = \frac{1}{2\cos\theta} = \frac{\sin\theta}{\sin 2\theta};$$

$$2\text{nd convergent} = \frac{2\cos\theta}{4\cos^2\theta-1} = \frac{\sin 2\theta}{\sin 3\theta};$$

. . . . . .

let $(n-1)$th and $(n-2)$th be

$$\frac{\sin(n-1)\theta}{\sin n\theta}, \qquad \frac{\sin(n-2)\theta}{\sin(n-1)\theta};$$

$$\therefore \quad n\text{th} = \frac{-2\cos\theta.\sin(n-1)\theta+\sin(n-2)\theta}{-2\cos\theta.\sin n\theta+\sin(n-1)\theta} = \frac{\sin n\theta}{\sin(n+1)\theta}.$$

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5. Shew that

$$\left\{\frac{1}{\cot\theta+}\ \frac{1}{\tan\theta+}\ \frac{1}{\cot\theta+}\cdots\right\}$$

$$\left\{\tan\theta+\frac{1}{\cot\theta+}\ \frac{1}{\tan\theta+}\ \frac{1}{\cot\theta+}\cdots\right\}=\tan^2\theta.$$

$$x=\frac{1}{\cot\theta+}\ \frac{1}{\tan\theta+x}=\frac{\tan\theta+x}{2+x\cot\theta};$$

$$\therefore\quad x\tan\theta+x^2=x\{\tan\theta+x\}=\tan^2\theta.$$

6. If AA_1, BB_1, CC_1 be arcs of great circles bisecting the angles of a spherical triangle, and meeting the opposite sides in A_1, B_1, C_1, then

$$\frac{\cos B-\cos C}{\cos\frac{A}{2}}\cot AA_1+\frac{\cos C-\cos A}{\cos\frac{B}{2}}\cot BB_1$$

$$+\frac{\cos A-\cos B}{\cos\frac{C}{2}}\cot CC_1=0.$$

$$\cot AA_1\sin c=\cot B\sin\frac{A}{2}+\cos c\cos\frac{A}{2}$$

$$=\cot B\sin\frac{A}{2}+\frac{\{\cos C+\cos A\cos B\}\cos\frac{A}{2}}{\sin A\sin B}=\frac{(\cos B+\cos C)\cos\frac{A}{2}}{\sin A\sin B};$$

$$\therefore\quad\frac{\cos B-\cos C}{\cos\frac{A}{2}}\cot AA_1=\frac{\cos^2B-\cos^2C}{\sin c\sin A\sin B};$$

$\therefore$ given expression

$$=\frac{\sin C}{\sin c}\cdot\frac{\cos^2B-\cos^2C+\cos^2C-\cos^2A+\cos^2A-\cos^2B}{\sin A\sin B\sin C}=0.$$

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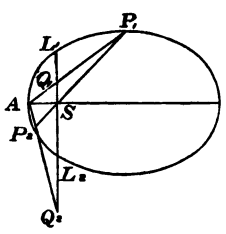
7. P_1SP_2 is a focal chord of a conic of which S is the focus and A the vertex; AP_1, AP_2 meet the latus rectum in Q_1, Q_2; then half the latus rectum is a mean proportional between SQ_1 and SQ_2.



Let $\angle ASP_1 = a$,

then equation of AP_1 is

$$\frac{l}{r} = e\cos\theta + \sec\frac{a}{2}\cos\left(\theta - \frac{a}{2}\right);$$

put $\theta = \frac{\pi}{2}$;

$$\therefore \quad SQ_1 = l\cot\frac{a}{2};$$

so $$SQ_2 = l\cot\frac{\pi+a}{2};$$

$$\therefore \quad SQ_1.SQ_2 = l^2 \text{ (regarding magnitude only).}$$

Or thus—

Lines drawn from any point on a conic to the extremities of chords that all pass through a fixed point, form a pencil in involution.

Thus we have an involution on the latus rectum, of which S is the centre; and Q_1, Q_2; L_1, L_2 are pairs of conjugate points.

$$\therefore \quad SQ_1.SQ_2 = SL_1.SL_2.$$

8. If c_1, c_2 be the lengths of the normal chords at the extremities of a focal chord of a parabola of which l is the latus rectum, then $(c_1^{-2} + c_2^{-2})^2 = (lc_1c_2)^{-2}$.

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\frac{x-x'}{l} = \frac{y-y'}{m} = r \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

substituting from (2) in (1) to find the length of any chord

$$m^2r^2 + 2(my' - 2al)r + y'^2 - 4ax' = 0;$$

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if $(x' y')$ is middle point of chord,

$$my' - 2al = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

and $\left\{\frac{1}{2} \text{ chord}\right\}^2 = -\frac{y'^2 - 4ax'}{m^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$

Equation of normal at (x_1, y_1) is

$$x.y_1 + y.2a - (x_1 + 2a)\, y_1 = 0;$$

comparing this with (2),

$$\frac{m}{y_1} = \frac{-l}{2a} = \frac{x'm - y'l}{y_1(x_1 + 2a)};$$

from these, and equations (1) and (3) we obtain

$$y' = \frac{-4a^2}{y_1}; \qquad x' = \frac{4a(x_1^2 + 2ax_1 + 2a^2)}{y_1^2};$$

$$\therefore \quad y'^2 - 4ax' = -\frac{16a^2(x_1 + a)^2}{y_1^2};$$

$$\text{also} \qquad \frac{l^2}{4a^2} = \frac{m^2}{y_1^2} = \frac{1}{4a(x_1 + a)}.$$

$$\therefore \quad \text{from (4)}, \qquad c_1^2 = \frac{16a(x_1 + a)^3}{x_1^2},$$

$$\text{so} \qquad c_2^2 = \frac{16a(x_2 + a)^3}{x_2^2} = \frac{16(x_1 + a)^3}{x_1}$$

since $x_1 x_2 = a^2$ for the extremities of a focal chord;

$$\therefore \quad c_1^{-2} + c_2^{-2} = \frac{x_1}{16a(x_1 + a)^2},$$

$$\text{and} \quad c_1^{-2} c_2^{-2} = \frac{x_1^3}{256a(x_1 + a)^6} = 16a^2.\{c_1^{-2} + c_2^{-2}\}^3;$$

$$\therefore \quad \{c_1^{-2} + c_2^{-2}\}^3 = \{4ac_1c_2\}^{-2}.$$

9. Tangents at points of the cissoid,

$$y^2 = \frac{x^3}{2a - x},$$

which subtend a right angle at the origin, intersect in a circle.

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The line $y = mx$ meets the curve in the point

$$x' = \frac{2am^2}{1+m^2}, \qquad y' = \frac{2am^3}{1+m^2};$$

equation of the tangent at this point is

$$(3m+m^3)x - 2y - 2am^3 = 0,$$

or $$(x-2a)m^3 + 3xm - 2y = 0,$$

an equation giving the values of m for the three lines drawn from the origin to the points of contact of tangents from any point (x, y);

if $m_1 m_2 = -1$, we easily obtain

$$2x^2+2y^2-5ax+2a^2 = 0$$

the equation of a circle.

11. A straight line moves between two fixed straight lines at right angles to each other; if the middle point of the straight line lie on the curve $\left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 = 1$, then the straight line will always touch an ellipse.

If $(x_1 y_1)$ is the middle point,

equation of the line is $\frac{x}{x_1} + \frac{y}{y_1} = 2$.....(1),

also equation of the ellipse, if it exists, must be of the form

$$Ax^2+By^2 = C.....(2);$$

tangent at any point

$$Axx'+Byy' = C;$$

this will coincide with (1), if

$$\frac{Ax'}{\frac{1}{x_1}} = \frac{By'}{\frac{1}{y_1}} = \frac{C}{2},$$

or $$\frac{Aax'}{\frac{a}{x_1}} = \frac{Bby'}{\frac{b}{y_1}} = \frac{C}{\frac{2}{1}} = \frac{\{A^2a^2x'^2+B^2b^2y'^2\}^{\frac{1}{2}}}{1};$$

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$$\therefore \quad A^2a^2x'^2+B^2b^2y'^2 \text{ must } = \frac{C^2}{4};$$

comparing this with (2), $Aa^2 = Bb^2 = \frac{C}{4}$;

$\therefore$ equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 4.$

12. If α, β, γ be the angles subtended at any point by the sides of a triangle, find the locus represented by

$$\cos\alpha+\cos\beta+\cos\gamma = -1.$$

$$\cos\alpha+\cos\beta+\cos(\alpha+\beta) = -1;$$

$$\therefore \quad \cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos^2\frac{\alpha+\beta}{2} = 0;$$

$$\therefore \quad \cos\frac{\alpha+\beta}{2} = 0; \quad \text{or} \quad \cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2} = 0;$$

whence $\alpha+\beta = \pi$, or $\alpha = \pi$, or $\beta = \pi$;

thus the locus is the three sides of the triangle.

13. A rectangular piece of paper $ABCD$ is turned over so that A coincides with C, and B with D, and is then flattened out. Shew that it is possible that the figure so formed may be a regular hexagon; and determine the condition that this should be so.

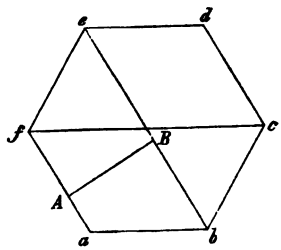
Width of paper $= AB$,

and length

$$= Bb+bc+cf+fA$$
$$= 3\sqrt{3}.AB,$$

since all the angles of the hexagon

$$= \frac{2\pi}{3}.$$



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17. Two parabolas can be drawn through four points on a circle; shew that they have their axes at right angles to one another, and parallel to the bisectors of the angles contained by any pair of chords through the four points.

If a conic and circle intersect, the chords of intersection make equal angles with the axis of the conic;

hence the problem.

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